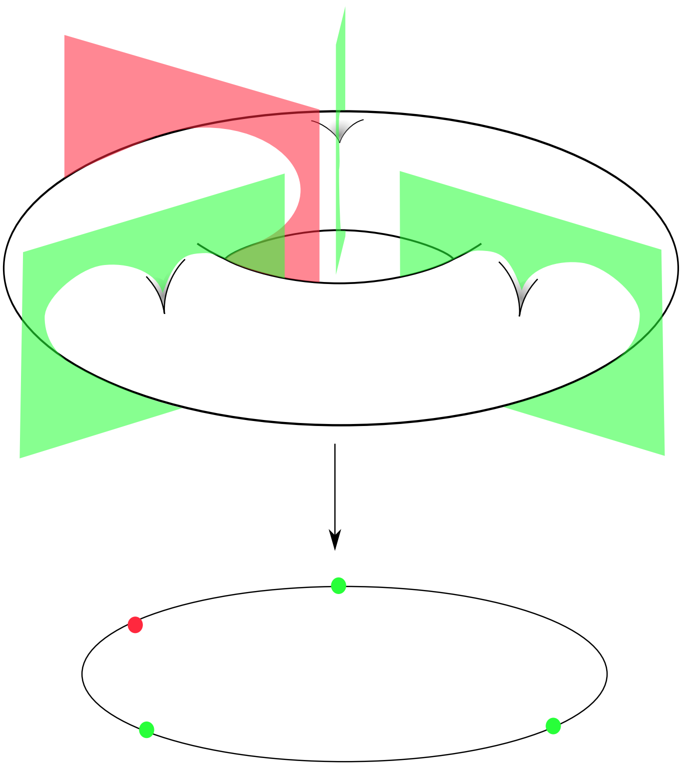
# SYMPLECTIC FIELD THEORY AND INTEGRABLE HIERARCHIES

TIMO KLUCK



Universiteit van Amsterdam

## MASTER THESIS

Title: Symplectic field theory and integrable hierarchies
Author: Timo Kluck
Thesis advisor: dr. S.V. Shadrin
Date: 23 August 2010

Cover illustration: a half-dimensional illustration of $\overline{\mathcal{M}}_{0,5}(\{\text{pt}\})$ as a fibre bundle over $\overline{\mathcal{M}}_{0,4}(\{\text{pt}\})$; see page 29.

Korteweg-de Vries Institute for Mathematics
Universiteit van Amsterdam
Floor 4C, Science Park 904, 1098 XH Amsterdam
The Netherlands
http://www.science.uva.nl/math

# CONTENTS

# INTRODUCTION

Symplectic field theory (SFT) is concerned with certain maps from Riemann surfaces into a specific type of manifolds, namely symplectic cobordisms. 'Counting' these curves and organising the result in a generating series gives an element $\mathbf{H}$ in an algebra of power series. The description of the limits of sequences of curves gives the (graded) commutation relation $[\mathbf{H}, \mathbf{H}] = 0$. This allows us to derive a commuting system of partial differential equations in this algebra. This is called an integrable hierarchy. As it turns out, the hierarchies we obtain from the simplest examples correspond already to interesting physical systems.

This thesis can be considered as a narrow introduction to Symplectic Field Theory as introduced by Eliashberg, Givental and Hofer in their paper [2], together with an overview of (a subset of) the work done by Fabert and Rossi to link this to integrable hierarchies [4, 3, 11].

The subject of SFT is so much work in progress that a very fundamental analytical result in SFT ("tranversality" of a certain "Cauchy-Riemann operator") has not yet been proved. It is common practise in SFT papers to state everything that relies on it nevertheless as a theorem. I will follow this convention. To give the reader (and myself) a taste of the kind of analysis involved, I will give an exposition of a related subject, called Floer homology, and work out (some of) the analysis there.

The outline of this thesis is then as follows.

In chapter 1, we will introduce Morse homology, which is a way of calculating the homology of a closed manifold in terms of the critical points of a (generic) real valued function. Conversely, it is a way of bounding the number of critical points of a function by using homology.

Next, we will discuss Floer homology in chapter 2. This can be considered a form of Morse homology, but in the infinite-dimensional setting of a loop space of a manifold. In this case, both the geometrical ideas involved and the necessary analysis become more complex.

These geometrical ideas are then useful background knowledge to study the geometry behind Symplectic Field Theory, which is chapter 3. Then comes a translation of these ideas into algebra in chapter 4.

We then discuss gravitational descendants in the well-known case of Gromov-Witten theory, and the current work in extending them to symplectic field theory. This comprises chapter 5. In the Gromov-Witten case, it is a well-known fact that integrable hierarchies are obtained; we discuss this in section 6.3. The same is found in symplectic field theory; however it is still very difficult to compute these hierarchies because the necessary tools, in particular a generalisation of the *topological recursion relations*, have not yet been developed.

However, the fact itself that symplectic field theory leads to integrable systems already gives new proofs of the structure of the integrable system in the Gromov-Witten case. This is discussed in the last sections 6.4 and 6.5.

ACKNOWLEDGEMENTS

I would like to thank my advisor Sergey Shadrin for his clear and intuitive explanations (which he would patiently repeat when needed), and also for introducing me to a topic as interesting and unexplored as symplectic field theory.

# Part I

# FLOER HOMOLOGY

# 1

## MORSE HOMOLOGY

### 1.1 GRADIENT FLOW

Let $M$ be a real, $n$-dimensional, closed manifold and $f$ a smooth function $f\colon M \to \mathbf{R}$. A point $x \in M$ is called *critical* for $f$ if $\mathrm{d}f$ vanishes at $x$. The goal of this first chapter is to sketch a proof of the following lower bound on the number of critical points of a generic $f$ in terms of the Betti numbers $h_i = \operatorname{rank} H_i(M, \mathbf{Z}/2\mathbf{Z})$:

$$\#\mathrm{crit}(f) \geq \sum_{i=0}^{n} h_i$$

Along the way, we will see general ideas that will be helpful in our discussion of Floer homology and Symplectic Field Theory.

The idea of the proof is that we can define a chain complex generated by the critical points of $f$. This chain complex turns out to be chain homotopic to the singular chain. This means that there must be a sufficient number of critical points to generate singular homology. But this is just the formula given above. We borrow liberally from the exposition in [7].

First of all, we choose a Riemannian metric $g$ on $M$. This allows us to define the *gradient* of $f$, with respect to $g$, written as $\nabla_g f$. For this, we see $g$ as a map $g\colon TM \to T^\vee M$ and define

$$\nabla_g f := g^{-1}(\mathrm{d}f)\,.$$

So the gradient of $f$ is a vector field on $M$. We will write $\Phi\colon M \times \mathbf{R} \to M$ or $\Phi_t\colon M \to M$ for its *negative* flow. Then a flow line is a path $\gamma_x(t) = \Phi_t(x)$ for a fixed starting point $x$. In this language, the flow is uniquely characterised by requiring that $\gamma_x(0) = x$ and $\mathrm{d}\gamma|_t(\frac{\partial}{\partial t}) = \nabla_g f|_{\gamma(t)}$ for all flow lines $\gamma$.

Now $g$ is positive-definite, which means that $f$ is strictly decreasing along flow lines of $-\nabla_g f$. Then we can be sure that $\gamma$ is injective, or in other words, that the flow $\Phi$ has no periodic orbits.

Since we assumed $M$ to be closed, we can also be sure that the limits $x^\pm := \lim_{t\to\pm\infty} \gamma(t)$ exist. It is clear that the gradient of $f$ must vanish in the limits

$t \to \pm\infty$, so these limits are critical points of $f$. We say that $\gamma$ is a flow line from $x^-$ to $x^+$. The number of flow lines between a given start- and end-point allows us to calculate $M$'s homology. The rest of this chapter will deal with this relation.

At a critical point $x$ we define the *Hessian* $\partial^2 f$ of $f$ as a map

$$\begin{aligned} \partial^2 f\colon T_x M &\to T_x^\vee M \\ \xi &\mapsto \nabla_\xi(\mathrm{d}f)\,, \end{aligned}$$

where $\nabla$ is some chosen connection. (The result does not depend on this choice.) We call a critical point *non-degenerate* if $\partial^2 f$ is. A function $f$ is called a *Morse function* if all its critical points are non-degenerate. In local coordinates $\partial^2 f$ is symmetric and can be diagonalised. The number of negative eigenvalues is the *Morse index* $i(x)$ of the critical point $x$.

Given two critical points $x^\pm$, their Morse indices $i(x^-)$ and $i(x^+)$ contain information about the existence and number of gradient flow lines from $x^-$ to $x^+$. Let us write $\tilde{\mathcal{M}}(x^-, x^+)$ for the set of $x \in M$ such that $\gamma_x$ is a flow line from $x^-$ to $x^+$. We have an straightforward $\mathbf{R}$-action on this set by translation: $c \in \mathbf{R}$ acts by sending $x \in \tilde{\mathcal{M}}(x^-, x^+)$ to $\gamma_x(c)$. We write

$$\mathcal{M}(x^-, x^+) := \tilde{\mathcal{M}}(x^-, x^+)/\mathbf{R}$$

which gives us a moduli-space of gradient flow lines. For a *generic* pair $(f, g)$ it can be shown to be a real manifold, with dimension given by

$$\dim \mathcal{M}(x^-, x^+) = i(x^-) - i(x^+) - 1\,. \tag{1.1}$$

In the generic case where this holds for all critical points $x^\pm$, we say that the pair $(f, g)$ is *Morse-Smale.* We will assume that it is in the rest of this chapter.

## 1.2 BROKEN FLOW LINES

We are going to define a chain complex and homology based on critical points and their Morse indices. To avoid orientation and sign issues, we choose coefficients for homology in $\mathbf{Z}/2\mathbf{Z}$. With more work, we can do the same with coefficients in $\mathbf{Z}$.

We define the Morse chain complex as follows. Let $C_i$ be the vector space over $\mathbf{Z}/2\mathbf{Z}$ generated by the critical points of Morse index $i$. We define a boundary operator $d\colon C_i \to C_{i-1}$ given by:

$$d(x) \;=\; \sum_{y \in C^{i-1}} c_{x,y}\, y \qquad \text{where } c_{x,y} = \#\mathcal{M}(x,y) \pmod 2$$

We want to prove that $d^2 = 0$. Since

$$d^2(x) = \sum_{\substack{z \in C^{i-2} \\ y \in C^{i-1}}} c_{y,z} c_{x,y} z$$

it is sufficient to prove that for every $x \in C_i$ and $z \in C_{i-2}$, the sum $\sum_y c_{y,z} c_{x,y}$ is zero. In other words, that the sum

$$\sum_y \#\mathcal{M}(x,y) \cdot \#\mathcal{M}(y,z)$$

is even. We do this by proving that the set

$$\bigcup_y \mathcal{M}(x,y) \times \mathcal{M}(y,z)$$

can be seen as the 0-dimensional "boundary" of (a suitable compactification of) the one-dimensional moduli space $\mathcal{M}(x,z)$. Such a boundary clearly has an even number of points[1].

So let us define this compactification of $\mathcal{M}(x,z)$. We do this by, for every sequence $(\gamma_i)$ in $\mathcal{M}(x,z)$, defining a limit for some sub-sequence. This limit will usually be another flow line $\gamma \in \mathcal{M}(x,z)$. Formally, this means that there are representatives $\tilde{\gamma}_i$ of $\gamma_i$ and a representative $\tilde{\gamma}$ of $\gamma$ such that $\lim_{i\to\infty} \tilde{\gamma}_i(t) = \tilde{\gamma}(t)$ for all $t$, and the convergence is uniform on compact subsets. In some special cases however, the limit is a *broken flow line*:

**Definition 1.2.1.** A sequence of flow lines $(\gamma_i) \in \mathcal{M}(x,z)$ converges to a *broken flow line* $(\gamma^1, \gamma^2) \in \mathcal{M}(x,y) \times \mathcal{M}(y,z)$ if

1. for some representatives $(\tilde{\beta}_i)$ of $(\gamma_i)$, the points $\big(\tilde{\beta}_i(0)\big)$ converge to the critical point $y \in M$;
2. for some other representatives $(\tilde{\alpha}_i)$, and for some representative $\tilde{\gamma}^1$ of $\gamma^1$, we have that $(\tilde{\alpha}_i(t))$ converges (uniformly on compact subsets) to $\tilde{\gamma}^1(t)$ for all $t$;
3. for some other representatives $\tilde{\delta}_i$, and for some representative $\tilde{\gamma}^2$ of $\gamma^2$, we have that $(\tilde{\delta}_i(t))$ converges (uniformly on compact subsets) to $\tilde{\gamma}^2(t)$ for all $t$.

**Proposition 1.2.2.** *With $x, z$ as above, every sequence $(\gamma_i)$ in $\mathcal{M}(x,z)$ has a sub-sequence converging to either a flow line in $\mathcal{M}(x,z)$ or to a broken flow line in $\mathcal{M}(x,y) \times \mathcal{M}(y,z)$ for some $y$.*

1 Note that if we can coherently orient the moduli spaces, then these boundary points come in pairs with opposite orientation. This suggests that if we choose the coefficients $c_{x,y}$ with appropriate signs, we can use coefficients in $\mathbf{Z}$. This can indeed be done, as can be seen (again) in [7].

*Proof.* Let $(\gamma_i)$ be any sequence of flow lines in $\mathcal{M}(x,z)$. We pick a neighbourhood $V$ of $x$ such that $\overline{V}$ contains no other critical points. Pick representatives $\tilde{\alpha}_i$ such that $w_i = \tilde{\gamma}_i(0)$ is the first point on the flow line such that $\gamma_i(t) \in \partial V$. Then the sequence of points $(w_i)$ in $\tilde{\mathcal{M}}$ has a sub-sequence with a limit $w$ in the compact closure $\overline{\tilde{\mathcal{M}}}$. We still write $(w_i)$ and $(\gamma_i)$ for this sub-sequence. We have $w \in \partial V$ so $w$ is not critical. Let $\tilde{\gamma}^1$ be the flow line starting at $w$ and $\gamma^1$ its equivalence class. Then we know the following:

1. The sequence $(\gamma_i)$ has representatives $\tilde{\alpha}_i$ such that $\lim_{i\to\infty} \tilde{\alpha}(t) = \tilde{\gamma}^1(t)$ (by definition).

This means that if $\gamma^1 \in \mathcal{M}(x,z)$, we have found a limit in $\mathcal{M}(x,z)$ and we are done. So suppose that it is not, so $w \notin \tilde{\mathcal{M}}$. Then $w \in \partial\tilde{\mathcal{M}}$.

2. The limit $\lim_{t\to-\infty} \tilde{\gamma}^1(t)$ is equal to $x$. It has to be a critical point in $V$ and $x$ is the only one.

3. We just assumed $\gamma^1 \notin \mathcal{M}(x,z)$, so we have that $\gamma^1 \in \mathcal{M}(x,y)$ for some critical point $y$. So $\mathcal{M}(x,y)$ is nonempty which means $i(y) < i(x)$.

Pick a decreasing sequence $U_i$ that is a neighbourhood basis of $y$ and pick $T_i$ large enough such that $\gamma^1(T_i) \in U_i$. Now $\tilde{\alpha}_j(T_i)$ converges to $\gamma^1(T_i)$ as $j \to \infty$, so we can pick $N_i$ such that if $j \geq N_i$ we have $\tilde{\alpha}_j(T_i) \in U_i$. Then the sequence $\tilde{\beta}_i(t) = \tilde{\alpha}_{N_i}(T_i + t)$ satisfies $\lim_{i\to\infty} \tilde{\beta}_i(0) = y$. We replace $(\gamma_i)$ by the sub-sequence $(\gamma_{N_i})$.

4. Then $\tilde{\beta}_i$ is a sequence of representatives for $\gamma_i$ satisfying $\lim_{i\to\infty} \tilde{\beta}_i(0) = y$.

Next, pick a neighbourhood $U$ of $y$ containing no other critical points and let $S_i$ be such that $\tilde{\beta}_i(S_i) \in \partial U$ is the first intersection of the flow $\tilde{\beta}_i$ with $\partial U$. Define $\tilde{\delta}_i(t) := \tilde{\beta}_i(S_i + t)$ and let $w'$ be the limit of (some sub-sequence of) $\tilde{\delta}_i(0)$. Then $w' \in \partial U$. Let $\tilde{\gamma}^2$ be the flow starting at $w'$ and $\gamma^2$ its equivalence class.

5. Then $\lim_{i\to\infty} \tilde{\delta}_i(t) = \tilde{\gamma}^2(t)$.

We just need to see that $\gamma^2 \in \mathcal{M}(y,z)$. Let $U_i$ be as above. Then there is $N_i$ such that $j \geq N_i$ implies $\tilde{\beta}_j(0) \in U_i$. In other words, $\tilde{\delta}_j(-S_j) \in U_i$. Fix some $S > 0$. Then $\tilde{\gamma}^2(-S) = \lim_{i\to\infty} \tilde{\delta}_i(-S) \in U$. So $\lim_{t\to-\infty} \tilde{\gamma}^2(t)$ is a critical point in $U$, so it is equal $y$. So we have $\gamma^2 \in \mathcal{M}(y,u)$ for some $u$. Now the moduli space $\mathcal{M}(y,u)$ is non-empty so $i(u) < i(y)$. So $i(u) \leq i(x) - 2$. The only such critical point in $\overline{\tilde{\mathcal{M}}}$ is $z$. So $u = z$ and $\gamma^2 \in \mathcal{M}(y,z)$.

This concludes the proof of the existence of point-wise limits. We omit proving the uniformity statement. $\square$

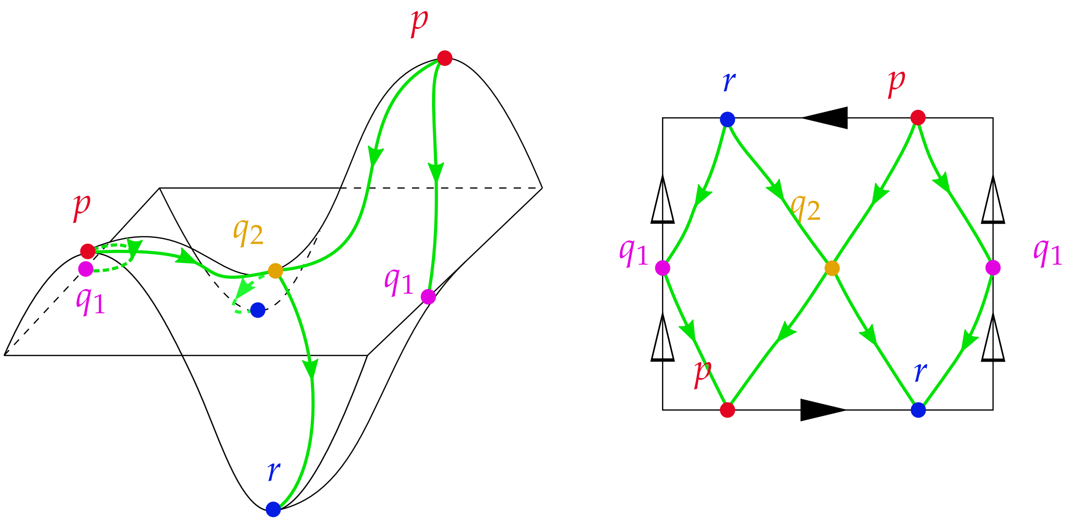



Figure 1.1: The graph of $f$ on the Klein bottle $K$ (depicted as a square with opposing edges suitably identified).

**Corollary 1.2.3.** *We have* $d^2 = 0$*, and therefore the pair* $(C_\bullet, d)$ *is a chain complex.* □

We define the *Morse homology* $H_\bullet^{\text{Morse}}(M, f, g)$ as the homology of this chain complex. Note that, as far as we can tell now, it depends equally well on $M$, $f$ and $g$. Let us, as an example, calculate a Morse homology of the Klein bottle. We will realise the Klein bottle $K$ as $[0, 2\pi] \times [0, \pi]$ where we identify all points $(0, y)$ with $(2\pi, y)$, and all points $(x, 0)$ with $(2\pi - x, \pi)$. Let $g$ be the flat metric, and define a function

$$f(x, y) := \sin(x)\cos(y)\,.$$

It is a smooth function on $K$. Its critical points are a point $p = (\frac{\pi}{2}, 0)$ of index 2 (a local maximum), two points $q_1 = (0, \frac{\pi}{2})$ and $q_2 = (\pi, \frac{\pi}{2})$ of index 1 (saddle points), and a point $r = (\frac{3\pi}{2}, 0)$ of index 0 (a local minimum). So $C_2 = \langle p \rangle$, $C_1 = \langle q_1, q_2 \rangle$ and $C_0 = \langle r \rangle$. This is all illustrated in figure 1.1.

There are two gradient flow lines from $p$ to each $q_i$, and two flow lines from each $q_i$ to $r$. So the differential is identically zero. Then we obtain

$$\begin{aligned} H_2^{\text{Morse}}(K, f, g) &= \langle [p] \rangle \\ H_1^{\text{Morse}}(K, f, g) &= \langle [q_1], [q_2] \rangle \\ H_1^{\text{Morse}}(K, f, g) &= \langle [r] \rangle\,. \end{aligned}$$

This agrees with singular homology (with coefficients in $\mathbf{Z}/2\mathbf{Z}$).

If we would have worked out the orientation issues, we would have found that the two gradient lines in $\mathcal{M}(p, q_i)$ have the same sign, and the two gradient lines in $\mathcal{M}(q_i, r)$ have opposing signs. This gives the differential $d(p) = 2q_1 + 2q_2$ and $d(q_i) = 0$. Then we get

$$\begin{aligned} H_2^{\text{Morse}}(K, f, g; \mathbf{Z}) &= \{0\} \\ H_1^{\text{Morse}}(K, f, g; \mathbf{Z}) &= \langle [q_1 + q_2] \rangle / 2 \langle [q_1 + q_2] \rangle + \langle [q_1] \rangle \cong \mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z} \\ H_1^{\text{Morse}}(K, f, g; \mathbf{Z}) &= \langle [r] \rangle \cong \mathbf{Z}\,. \end{aligned}$$

Again, this agrees with the (possibly more familiar) singular homology with coefficients in $\mathbf{Z}$. This is a general fact:

**Theorem 1.2.4.** *The chain complex $C_\bullet^{\text{Morse}}(M, f, g)$ is chain homotopic to the singular chain $C_\bullet^{\Delta}(M)$ for any Morse-Smale[2] pair $(f, g)$. In particular, their homologies $H_\bullet^{\text{Morse}}(M, f, g)$ and $H_\bullet(M)$ are isomorphic.*

*Proof.* This is proved in [7]. □

In particular, we see that Morse homology does not depend on the choice of the pair $(f, g)$, as long as it is generic.

## 1.3 INDEPENDENCE OF THE CHOICE OF $(f, g)$

By theorem 1.2.4, we already know that Morse homology $H_\bullet^{\text{Morse}}(M, f, g)$ does not depend on the choice of the function $f$ and metric $g$, as long as they are generic. Since we only gave a reference for its proof, we may wonder whether we can show this invariance in a direct way.

In fact, we can. Let us start with two given pairs $(f_0, g_0)$ and $(f_1, g_1)$. Let $(f_t, g_t)$ be any smooth homotopy between them. We consider the manifold $M \times I$ and a vector field on it, given by:

$$X(x, t) = \nabla_{g_t} f_t(x) + -t(t-1)\frac{\partial}{\partial t}$$

for $(x, t) \in M \times I$. It is clear that this vector field has a-periodic flow and that its critical points are given by

$$\text{crit}(X) = \text{crit}(f_0) \times \{0\} \cup \text{crit}(f_1) \times \{1\}$$

2 This probably depends on the stricter definition of Morse-Smale as alluded to in an earlier footnote.

We can calculate moduli spaces of flow lines of $X$ between these critical points. We find Morse indices

$$\begin{aligned} i_{M\times I}(x \times \{0\}) &= i_M(x) + 1 \\ i_{M\times I}(x \times \{1\}) &= i_M(x) . \end{aligned}$$

Under genericity conditions for the homotopy $(f_t, g_t)$ similar to those for a pair $(f, g)$, we find that the moduli spaces of $X$-flow lines between critical points are smooth manifolds whose dimension is given by equation (1.1). This means that if $x \in M$ has index $i$ for $(f_0, g_0)$ and $y \in M$ has the same index $i$ for $(f_1, g_1)$, then they have indices $i + 1$ and $i$ respectively for the flow of $X$, so the moduli space

$$\tilde{\mathcal{M}}((x, 0), (y, 1))$$

is zero-dimensional. Then it makes sense to define maps

$$\begin{aligned} \phi_i \colon C_i(M, f_0, g_0) &\to C_i(M, f_1, g_1) \\ x &\mapsto \sum_{\substack{y\in \mathrm{crit}(f_1) \\ i(y)=i}} \#\mathcal{M}((x, 0), (y, 1)) \end{aligned}$$

We are trying to show that this induces an isomorphism on homology. We will need several steps for this:

1. First, we want that $\phi$ is a chain map (i.e. $\phi_i \circ d = d \circ \phi_{i+1}$).
2. Then, we want that if $\Psi$ and $\Phi$ are two homotopic homotopies between the same maps and metrics, then they induce chain homotopic chain maps $\psi$ and $\phi$.
3. Next, if $\Psi$ and $\Phi$ are two homotopies with the endpoint of $\Psi$ agreeing with the start of $\Phi$, then we want that the chain map induced by the concatenation of homotopies $\Psi * \Phi$ is chain homotopic to the concatenation of chain maps $\phi \circ \psi$.
4. We apply 2 and 3 to a homotopy $\Psi$ and its reverse $\Phi$, to conclude that $\phi \circ \psi$ is chain homotopic to the identity. Then they induce an isomorphism on homology.

The first step is not very difficult. We have that

$$\begin{aligned} \phi_i \circ d(x) &= \sum_{\substack{z\in \mathrm{crit}(f_1) \\ i(z)=i}} \sum_{\substack{y\in \mathrm{crit}(f_0) \\ i(y)=i+1}} \#\mathcal{M}((y, 0), (z, 1)) \#\mathcal{M}((x, 0), (y, 0)) \cdot z \\ d \circ \phi_{i+1}(x) &= \sum_{\substack{z\in \mathrm{crit}(f_1) \\ i(z)=i}} \sum_{\substack{y\in \mathrm{crit}(f_1) \\ i(y)=i+1}} \#\mathcal{M}((y, 1), (z, 1)) \#\mathcal{M}((x, 0), (y, 1)) \cdot z \end{aligned}$$

We can interpret the summands as counting broken flow lines from $(x, 0)$ to $(z, 1)$. So they count boundary points of the compactified 1-dimensional moduli space $\mathcal{M}((x, 0), (z, 1))$. It has an even number of points, so $\phi_i \circ d(x) + d \circ \phi_{i+1}(x) = 0 \pmod 2$. This just means $\phi_i \circ d = d \circ \phi_{i+1}$. So $\phi$ is a chain map.

The proofs of 2 and 3 are similar, so we will only prove 2. We will construct a chain homotopy $K \colon C_i \to C_{i+1}$ such that $d \circ K + K \circ d = \psi - \phi$. We do that as follows. Consider the homotopy between the homotopies as a family $(f_\tau, g_\tau)$ where $\tau$ runs over a "digon" $D$, a square with two opposing edges each collapsed to a point. Define a (generic) vector field $\tilde{X}$ on $D$ which agrees with $X_\phi$ and $X_\psi$ on the edges, and which has a critical point of index 2 on one vertex and with index 0 on the other. Next, consider the vector field $\hat{X}$ on $D \times M$, given by $\tilde{X} + \nabla_{g_\tau} f_\tau$. Then we define $K$ as the map counting flow lines. This gives the chain homotopy equation. The proof of 3 is obtained by replacing $D$ with a triangle.

Applying 4 now gives that $H_\bullet^{\text{Morse}}(M, f, g)$ is an invariant for $M$.

# 2

# FLOER HOMOLOGY

## 2.1 HAMILTONIAN SYSTEM

SYMPLECTIC MANIFOLDS Let $V$ be a vector space and $\omega\colon V \otimes V \to \mathbf{R}$ be an anti-symmetric, non-degenerate pairing. Here, non-degenerate means that $\omega(x, \cdot) \equiv 0$ implies $x = 0$. Equivalently, it means that the matrix $\tilde{\omega}$ given by $\tilde{\omega}_{ij} = \omega(e_i \otimes e_j)$ for a given basis $(e_i)_i$ of $V$ has nonzero determinant. Such a pairing can only exist when $W$ is even dimensional, since

$$\det(\tilde{\omega}) = \det(\tilde{\omega}^T) = \det(-\tilde{\omega}) = (-1)^{\dim W} \det(\tilde{\omega}) \,.$$

The pair $(V, \omega)$ is called a *symplectic vector space*.

The canonical example is the following. Let $p_1, \cdots, p_n$ together with $q_1, \cdots, q_n$ be basis vectors for $\mathbf{R}^{2n}$. Consider the anti-symmetric pairing $\omega_0(\cdot, \cdot)$ given by

$$\omega_0(p_i, q_j) = -\omega_0(q_j, p_i) = \delta_{ij}$$

and zero on other combinations of basis vectors. Then $(\mathbf{R}^{2n}, \omega_0)$ is a symplectic vector space. In fact, it is not difficult to prove that any other $2n$-dimensional symplectic vector space is isomorphic to this one; i.e. there always exists a vector space isomorphism $\phi\colon V \to \mathbf{R}^{2n}$ such that $\phi^*\omega_0 = \omega$. We define the *symplectic group* $\mathrm{Sp}(n) \subseteq \mathrm{Gl}(2n)$ of transformations that leave $\omega_0$ invariant.

A *symplectic manifold* is a pair $(W, \omega)$ with $W$ a real smooth manifold and $\omega$ a closed two-form on $W$, such that $(T_x W, \omega_x)$ is a symplectic vector space for all $x \in W$. A standard example for a symplectic manifold is the cotangent bundle $T^\vee M$ to a given manifold $M$, with coordinates $q_i$ on $M$, coordinates $p_i$ expressing the vector bundle fibres $T^\vee_q M$ on their basis $(\mathrm{d}q_i)$, and the symplectic form $\omega$ given by $\omega = \sum_i \mathrm{d}p_i \wedge \mathrm{d}q_i$. In fact, any symplectic manifold is locally of this type (this is the content of *Darboux's theorem*).

An *almost complex structure* on $W$ is is a smoothly changing linear map $J$ on tangent spaces $T_x M$ with $J^2 = -1$. It is a standard fact (see e.g. [9]) that we can choose, on any $W$, an almost complex structure $J$ such that $\omega(\cdot, J\cdot)$ is a Riemannian metric. For instance, in the example above, we can choose $J\frac{\partial}{\partial q_i} = -\frac{\partial}{\partial p_i}$ and $J\frac{\partial}{\partial p_i} = \frac{\partial}{\partial q_i}$.

HAMILTONIAN FLOW Let $(W, \omega)$ be a symplectic manifold and $H_t = H_{t+1}$ a time-dependent smooth function on $W$. Let $X_t$ be the time dependent vector field given by $\omega^{-1}(\mathrm{d}H_t)$ (here we identify $\omega$ with the map $\xi \in T_xW \mapsto \omega(\xi, \cdot) \in T_x^\vee W$). Such a vector field is called *Hamiltonian*. We are interested in its periodic orbits. In other words, if we write $\Phi_t \colon W \to W$ for the flow of $X_t$, then the fixed points $y$ of $\Phi_1$ correspond one-on-one to solutions $x \colon S^1 \to W$ of $\mathrm{d}x(\frac{\partial}{\partial t}) = X_t|_{x(t)}$. We call such a solution *non-degenerate* if $\det(\mathrm{d}\Phi_1|_y - \mathrm{id}) \neq 0$.

FLOER HOMOLOGY Consider the *contractible loop space*

$$\mathcal{L}(W) := \{x \colon S^1 \to W \mid x \text{ is smooth and contractible}\}$$

We write $T_x\mathcal{L}(W)$ for the tangent space at $x \in \mathcal{L}$; in other words, a tangent vector $\xi \in T_x\mathcal{L}(W)$ is a map $\xi \colon S^1 \to TW$ with $\xi(t) \in T_{x(t)}W$. Then we can define a one form $\Psi$ given by (writing $\dot{x}(t)$ for $\mathrm{d}x(\frac{\partial}{\partial t})$)

$$\Psi_x(\xi) = \int_{S^1} \omega(\dot{x}(t) - X_t(x(t)), \xi(t))\,\mathrm{d}t$$

and we see that if $\Psi_x \equiv 0$, then $x$ is a periodic solution. We can try to apply the ideas of Morse homology in this situation, where $\mathcal{L}(W)$ takes the role of the manifold $M$, where $\Psi$ takes the role of $\mathrm{d}f$, and where $\omega(\cdot, J_t\cdot)$ takes the role of the metric $g$ for a periodic family of almost complex structures $J_t$ that are compatible with $\omega$. This is known as *Floer homology*.

To make all the details work, we will have to deal with the following complications. Let us start by noting that a flow line in $\mathcal{L}$ is a one-parameter family of loops in $W$, so it can be seen as a map from a cylinder into $W$.

First of all, we want the moduli spaces of flow lines to be smooth manifolds. This requires perturbing the Hamiltonian to be generic. Next, we want the moduli spaces to be compact. This requires adding analogues of broken flow lines to the moduli spaces. Furthermore, it requires dealing with the 'bubbling' phenomenon in a otherwise converging sequence of cylinders.

Secondly, we want to be able to assign indices to the periodic orbits that allow us to make a statement about the dimension of these moduli spaces. It turns out that it is only possible to assign *relative* indices for a pair of orbits. These indices will be the *Conley-Zehnder indices*.

CHERN CLASS AND MONOTONICITY These things are easier when we lay the requirement on $(W, \omega)$ that it be *monotone*, which means that

$$\int_{S^2} v^* c_1 = \tau \int_{S^2} v^* \omega \tag{2.1}$$

for every smooth map $v\colon S^2 \to W$ and for some $\tau > 0$. Here, $c_1$ is the first Chern class of the almost complex bundle $(TW, J_t)$, which does not depend on $J_t$. We also define the *minimal Chern number* of $(W, \omega)$ as the integer

$$N := \inf \left( \{ \int_{S^2} v^* c_1 \mid v\colon S^2 \to W \} \cap \mathbf{R}_{>0} \right)$$

When $\int_{S^2} v^* c_1 = 0$ for all $v$, then we define $N := \infty$, but this does not happen in the monotone case. In the monotone case, $N$ is nonzero.

We will assume 2.1 in the rest of this chapter. We will also normalise $\omega$ such that

$$\int_{S^2} v^* \omega \in \mathbf{Z} \tag{2.2}$$

for all smooth $v$.

The following treatment follows largely the exposition in [13].

## 2.2 CONNECTING CYLINDERS

Suppose that we have chosen $H_t$ such that all its periodic orbits are non-degenerate, and that we have fixed a family $J_t$ of $\omega$-compatible almost complex structures. Then the 1-form $\Psi$, and the metric that $\omega(\cdot, J_t \cdot)$ induces on $\mathcal{L}$, lead to the flow $Z$ given by (for $x \in \mathcal{L}$, and so $Z(x) \in T_x\mathcal{L}$)

$$Z(x)(t) \quad = \quad J_t(x(t))\dot{x}(t) - \nabla H_t(x(t))$$

where $\nabla$ is the gradient induced by $\omega(\cdot, J_t \cdot)$. We will write the negative flow lines of $Z$ as $u\colon s \in \mathbf{R} \mapsto u(s, \cdot) \in \mathcal{L}$, so $u$ is a function of two variables $s$ and $t$ and $u(s,t) = u(s, t+1)$. The above equation for $Z$ gives that $u$ should satisfy the partial differential equation

$$\frac{\partial u}{\partial s} + J_t(u)\frac{\partial u}{\partial t} - \nabla H_t(u) = 0\,. \tag{2.3}$$

We will define the *energy* of a solution $u$ as

$$E(u) := \frac{1}{2}\int_0^1 \int_{-\infty}^{\infty} \left( \left|\frac{\partial u}{\partial s}\right|^2 + \left|\frac{\partial u}{\partial t} - \nabla H_t(u)\right|^2 \right) \mathrm{d}s\,\mathrm{d}t$$

and we will consider only solutions of (2.3) that have finite energy. It turns out that these are exactly the ones that connect periodic orbits, so we will refer to them as connecting cylinders. More precisely:

**Theorem 2.2.1.** *Suppose $u$ solves (2.3). Then $E(u) < \infty$ if and only if there are periodic solutions $x^{\pm}(t)$ such that*

$$\lim_{s\to\pm\infty} u(s,t) = x^{\pm}(t) \tag{2.4}$$

*and $\lim_{s\to\pm\infty} \partial_s u(s,t) = 0$, where both limits are uniform over $t$. In this case we have*

$$|\frac{\partial u}{\partial s}| = \mathcal{O}(\mathrm{e}^{-s})$$

*Proof.* This is [12, prop. 4.2]. □

We will write $\mathcal{M}(x^-, x^+)$ for the space of all solutions to (2.3) and (2.4).

We can define the *action functional* as

$$a_H(x,u) := -\int_{D^2} u^*\omega - \int_0^1 H_t(x(t))\,\mathrm{d}t$$

where $u\colon D^2 = \{z \in \mathbf{C} \mid |z| \leq 1\} \to M$ is a chosen capping surface for $x$, which means that $u(\mathrm{e}^{2\pi i t}) = x(t)$. Because of (2.2), a different choice of $u$ will lead to an integer difference in $a_H(x)$. So we can also define $a_H(x)$, without reference to $u$, as a functional taking values in $\mathbf{R}/\mathbf{Z}$.

When $u \in \mathcal{M}(x^-, x^+)$, then we have

$$E(u) = a_H(x^-, u^-) - a_H(x^+, u^+) \tag{2.5}$$

where $u^-$ is any capping surface for $x^-$, and where $u^+$ agrees with $u^-\#u$.

## 2.3 PERTURBATION OF CONNECTING CYLINDERS

In order to study the moduli space $\mathcal{M}(x^-, x^+)$, we pick an element $u$ and study 'nearby' solutions. Let us re-write equation (2.3) as

$$\bar{\partial}_{H,J} u = 0$$

and we consider the map

$$\begin{aligned}\bar{\partial}_{H,J}\colon C^\infty(\mathbf{R}\times S^1, W) &\to C^\infty(\mathbf{R}\times S^1, TW)\\ u &\mapsto \bar{\partial}_{H,J}u \in C^\infty(\mathbf{R}\times S^1, T_uW)\end{aligned}$$

as a vector field on the space of solutions $u$ to (2.4). The map $\bar{\partial}_{H,J}$ is customarily referred to as the *Cauchy-Riemann operator*[1]. The Levi-Civita connection on $W$ induces (point-wise) a connection $\nabla$ on this space. Then, infinitesimally near a zero $u$, other zeroes are found in directions $\xi \in C^\infty(\mathbf{R}\times S^1, T_uW)$ for which $\nabla_\xi \bar{\partial}_{H,J} = 0$. One may suspect, therefore, that the dimension of the moduli space is given by the dimension of $\ker(\xi \mapsto \nabla_\xi \bar{\partial}_{H,J})$. To make this rigorous, and to establish that the moduli space is a smooth manifold, we should relate this infinitesimal picture to the 'real' local picture. This can be done (thanks to the Riemannian structure) using an exponential map, and details are in [8, § 3.3]. We will content ourselves here with the infinitesimal picture.

So we are interested in the kernel of the map

$$\begin{aligned}\xi &\mapsto \nabla_\xi\left(\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} - \nabla H(u)\right)\\ &= \nabla_s\xi + J(u)\nabla_t\xi + \left(\nabla_\xi J(u)\right)\frac{\partial u}{\partial t} - \nabla_\xi \nabla H(u)\end{aligned}$$

for which we will write $\tilde{D}$. We choose a trivialisation $T(s,t)\colon \mathbf{R}^{2n} \to u^*T_{u(s,t)}M$ of the symplectic bundle $u^*TM \to \mathbf{R}\times S^1$, which means that the standard structures $\omega_0$ and $J_0$ on $\mathbf{R}^{2n}$ map to $\omega$ and $J$. We write $D := T^{-1}\circ\tilde{D}\circ T$. Then we obtain

$$\begin{aligned}D\xi &= \partial_s\xi + J_0\partial_t\xi + T^{-1}\left(\nabla_s T + J(u)\nabla_t T + \nabla_T J(u)\frac{\partial u}{\partial t} - \nabla_T\nabla H(u)\right)\xi\\ &\quad (\text{here } \xi\colon \mathbf{R}\times S^1 \to \mathbf{R}^{2n})\\ &=: \partial_s\xi + J_0\partial_t\xi + S\xi \qquad (2.6)\end{aligned}$$

where $S(s,t)$ is a family of matrices. The limits $S^\pm$ for $s\to\pm\infty$ are given by

$$S^\pm(t)\xi = \lim_{s\to\pm\infty} S(s,t)\xi = \left(T^\pm\right)^{-1} J\left(\nabla_t T^\pm\xi - \nabla_{T^\pm\xi}X_H\right)$$

because $\nabla_t T \to 0$ and $\nabla_T(J\frac{\partial u}{\partial t} - \nabla_H) \mapsto \nabla_{T^\pm}X_H$ by (2.4). They are symmetric. We can therefore replace $D$ by $D+K$ such that $K$ is compact and such that the resulting operator can be expressed as in (2.6) with $S(s,t)$ symmetric for all $s,t$. This compact perturbation will have no effect for the dimension that we are going to calculate.

1 More precisely, $\partial_s + J\partial_t$ is the Cauchy-Riemann operator and we consider $\nabla H$ to be a perturbation.

## 2.4 DIMENSIONS OF SOLUTIONS TO PDE'S

FREDHOLM OPERATORS Suppose $L\colon A \to B$ is a bounded linear operator between Banach spaces. We call $L$ a *Fredholm operator* if it has closed range and if it has finite dimensional kernel and co-kernel (the co-kernel is $B/\overline{\operatorname{im} L}$). Its *Fredholm index* is $\dim \ker L - \dim \operatorname{coker} L$.

We call $K\colon A \to B$ *compact* if every bounded subset in $A$ has an image with compact closure in $B$. If $L$ is Fredholm, then so is $L + K$. In the operator norm, we have that Fredholm indices are stable under small perturbations.

If $f\colon X \to Y$ is a smooth function, then we call $f$ Fredholm if $\mathrm{d}f\colon T_xX \to T_{f(x)}Y$ is Fredholm for every $x \in X$. Because Fredholm indices are stable under perturbation, the index of $\mathrm{d}f$ does not depend on $x$ and we call it the Fredholm index of $f$. If $y \in Y$ is a point such that $\mathrm{d}f|_x$ is surjective for every $x \in f^{-1}(y)$, then we call $y$ a *regular value* of $f$. It is a theorem (the infinite dimensional analogue of the implicit function theorem) that in this case $f^{-1}(y)$ is a smooth, finite dimensional manifold. Clearly, the tangent space at $x$ to $f^{-1}(y)$ is given by $\ker \mathrm{d}f|_x$ and so, since $\mathrm{d}f|_x$ is onto, its dimension is the index of $f$.

SOBOLEV SPACES Let $C^\infty$ be the space of all smooth maps $\xi\colon \mathbf{R} \times S^1 \to \mathbf{R}^{2n}$ (where $\mathbf{R}^{2n}$ carries the standard symplectic, almost complex, and Riemannian structure) that satisfy $\|\xi(s,t)\| = \mathcal{O}(\mathrm{e}^{-|s|})$ and similarly for its partial derivatives. We will complete it into a Banach space in two ways. We define norms

$$\begin{aligned}
\|\xi\|^p_{L^p} &:= \int_{-\infty}^{\infty}\int_{S^1} \|\xi(s,t)\|^p \,\mathrm{d}t\,\mathrm{d}s \\
\|\xi\|^p_{W^{1,p}} &:= \int_{-\infty}^{\infty}\int_{S^1} \|\xi(s,t)\|^p + \|\partial_s\xi(s,t)\|^p + \|\partial_t\xi(s,t)\|^p \,\mathrm{d}t\,\mathrm{d}s
\end{aligned}$$

where $\|\cdot\|$ is the norm induced by $\omega(\cdot, J\cdot)$, and we define $L^p$ and $W^{1,p}$ as the completion of $C^\infty$ with respect to these norms. It is clear that $D$ as in (2.6) extends to an operator

$$D\colon W^{1,p} \to L^p\,.$$

One can prove that (under a non-degeneracy condition) this operator is Fredholm, and calculate its index. An argument called *elliptic regularity* asserts that for $p > 2$, any element in $W^{1,p}$ that is in the kernel of $D$ is actually smooth. This means that the index of $D$ is equal to the dimension of the moduli space of *smooth* connecting orbits.

Details of this approach can be found in [8, appendix B] and in [10, appendix B].

THE CONLEY-ZEHNDER INDEX Let $p_1, \cdots, p_n$ together with $q_1, \cdots, q_n$ be basis vectors for $\mathbf{R}^{2n}$ and $\omega_0$ the standard symplectic structure. Recall that the symplectic group $\mathrm{Sp}(n) \subseteq \mathrm{Gl}(2n)$ consists of all transformations that leave $\omega_0$ invariant. The unitary group $U(n)$ also acts on $\mathbf{R}^{2n}$ if we identify each direct summand $\langle p_i, q_i \rangle$ with a copy of $\mathbf{C}$. It is easy to see that in this way, $U(n)$ is a subgroup of $\mathrm{Sp}(n)$. In fact, $U(n)$ is a deformation retract of $\mathrm{Sp}(n)$.

Because of this, we can continuously extend the map $\det\colon U(n) \to S^1$ to a map $\rho\colon \mathrm{Sp}(n) \to S^1$. We do this in such a way that $\rho$ is multiplicative with respect to direct sums, invariant under similarity, and such that it takes the values $\pm 1$ on symplectic matrices with no eigenvalues on the unit circle.

We write $\mathrm{Sp}^*(n) \subseteq \mathrm{Sp}(n)$ for the open dense set of matrices that do not have 1 as an eigenvalue. It has two connected components, corresponding to positive and negative values of $\det(\mathrm{id} - \Psi)$. We fix two representing elements in these components, namely the matrices $A^+ = -\mathrm{id}$ and

$$A^- = \mathrm{diag}(2, -1, \cdots, -1, 1/2, -1, \cdots, -1)$$

For a path $\Phi\colon [0,1] \to \mathrm{Sp}(n)$ for which $\Phi(0) = \mathrm{id}$ and $\Phi(1) \in \mathrm{Sp}^*(n)$, we consider an extension $\hat{\Phi}\colon [0,2] \to \mathrm{Sp}(n)$ such that either $\hat{\Phi}(2) = A^+$ or $\hat{\Phi}(2) = A^-$, and such that $\det(\mathrm{id} - \hat{\Phi})$ does not change sign on $[1,2]$. Then $(\rho \circ \hat{\Phi})(2) = \pm 1$, so $(\rho^2 \circ \hat{\Phi})(2) = 1$ and therefore $\rho^2 \circ \hat{\Phi}$ can be regarded as a loop $S^1 \to S^1$. We define the *Conley-Zehnder index* $\mu_{\mathrm{CZ}}$ of the path $\Phi$ as $\mu_{\mathrm{CZ}} = \deg \rho^2 \circ \hat{\Phi}$. It is well-defined since $\hat{\Phi}$ is unique up to homotopy.

## 2.5 THE DIMENSION FORMULA

Let $u$ be a cylinder connecting $x^-$ with $x^+$. Let us fix a capping surface $v^-$ for $x^-$. We define the capping surface $v^+$ for $x^+$ as the connected sum of $u$ and $v^-$, written $v^- \# u$; more precisely, we define $v^+(z) = v^-(2z)$ for $|z| < 1/2$ and $v^+(re^{2\pi i t}) = u(\beta(r), t)$ for $r \geq 1/2$ and for some homeomorphism $\beta\colon [1/2, 1] \to [-\infty, \infty]$.

Since $D^2$ is contractible, the pullback symplectic bundle $(v^\pm)^* TW$ can be trivialised, say by $\phi\colon (v^\pm)^* TW \to D^2 \times \mathbf{R}^{2n}$. If we fix an identification of $(v^\pm)^* TW$ with $\mathbf{R}^{2n}$ on the point $x^\pm(0)$, then this trivialisation is unique up to homotopy. Parallel transport along the Hamiltonian flow $X_H$ defines a map $T_{x^\pm(0)} W \to T_{x^\pm(t)} W$ and so the trivialisation $\phi$ gives a map $\{1\} \times \mathbf{R}^{2n} \to \{e^{2\pi i t}\} \times \mathbf{R}^{2n}$. We see this as a path $\Phi\colon [0,1] \to \mathrm{Sp}(n)$. We define the Conley-Zehnder index of the pair $(x^\pm, v^\pm)$ as the Conley-Zehnder index of this path.

The Conley-Zehnder index depends on the choice of capping surface as follows:

$$\mu_{\mathrm{CZ}}(x, A\#v) = \mu_{\mathrm{CZ}}(x,v) - 2c_1(A) \qquad A \in \pi_2(W) \tag{2.7}$$

**Theorem 2.5.1.** *The index of the Fredholm operator $D$ is given by*

$$\operatorname{index} D = \mu_{\mathrm{CZ}}(x^-, v^-) - \mu_{\mathrm{CZ}}(x^+, v^-\#u) \tag{2.8}$$

Now let $\mathcal{H}_{\mathrm{reg}}$ be the set of periodic Hamiltonians $H$ such that all orbits are non-degenerate and such that for all $x^-, x^+, u$ the map $D$ has 0 as a regular value. Then every component (containing $u$) of every $\mathcal{M}(x^-, x^+)$ is a smooth manifold, and its dimension agrees with the Fredholm index.

We now refer to [13] (which itself refers to [6]) for the fact that $\mathcal{H}_{\mathrm{reg}}$ is a countable intersection of open dense sets. Let us write

$$\eta_H(x) := \mu_{\mathrm{CZ}}(x,u) - 2\tau a_H(x,u)$$

which is well-defined because of (2.5), (2.1), and (2.7). Then we obtain

**Corollary 2.5.2.** *For generic $H$, the moduli space $\mathcal{M}(x^-, x^+)$ has smooth connected components with the dimension of the component containing $u$ given by* (2.8)*, which is equal to*

$$\eta_H(x^-) - \eta_H(x^+) + 2\tau E(u) \tag{2.9}$$

□

## 2.6 COMPACTIFICATION OF THE MODULI SPACE

Consider three periodic orbits $x, y, z$ and two connecting cylinders $u_1 \in \mathcal{M}(x,y)$ and $u_2 \in \mathcal{M}(y,z)$. We want to consider this a broken flow line between $x$ and $z$. This means trying to find a family of connecting cylinders in $\mathcal{M}(x,z)$ that approximate $u_1$ followed by $u_2$.

We first try this by interpolation. Let $\xi_1(s,t) \in T_{y(t)}W$ be such that $\exp_{y(t)}(\xi_1(s,t)) = u_1(s,t)$ for large positive $s$, and similarly $\xi_2(s,t)$ such that $\exp_{y(t)}(\xi_2(s,t)) = u_2(s,t)$ for large negative $s$.

$$\tilde{v}_R(s,t) := \begin{cases} u_1(s+R,t) & \text{for } s < -\frac{1}{2}R \\ \exp_{y(t)}(\beta_R(s)\xi_1(s+R,t)) & \text{for } -\frac{1}{2}R \le s \le -\frac{1}{2}R + 1 \\ y(t) & \text{for } -\frac{1}{2}R + 1 \le s \le \frac{1}{2}R - 1 \\ \exp_{y(t)}(\beta_R(s)\xi_2(s-R,t)) & \text{for } \frac{1}{2}R - 1 \le s \le \frac{1}{2}R \\ u_2(s-R,t) & \text{for } \frac{1}{2}R < s \end{cases}$$

where $\beta_R$ is a smooth cut-off function with $\beta_R(s) = 0$ on $-\frac{1}{2}R + 1 \leq s \leq \frac{1}{2}R - 1$ and $\beta_R(s) = 1$ on $|s| \geq \frac{1}{2}R$. So we have smooth functions $\tilde{v}_R$ with limits $x$ and $z$. They will probably not, however, satisfy the Cauchy-Riemann equation (2.3) on the interpolation parts $\frac{1}{2}R - 1 \leq |s| \leq \frac{1}{2}R$. We will try to establish that a uniquely determined nearby solution $v_R$ exists.

For this, consider the Cauchy-Riemann operator and its covariant derivative $D_R := D_{\tilde{v}_R}$ on (exponentially decaying, smooth) vector fields along $\tilde{v}_R$. Suppose that (1) we know that the Cauchy-Riemann operator takes a 'sufficiently small' value on $\tilde{v}_R$ as $R \to \infty$; (2) we know that its covariant derivative is surjective; and (3) that this covariant derivative takes 'sufficiently large' values. Then in particular, it takes 'sufficiently large' values in the direction opposite to the 'sufficiently small' value of the Cauchy-Riemann operator. Consequently, it must have nearby zeroes for sufficiently large $R$. That such an approach can be made precise, can be seen in [8, appendix A]. It is also easily generalised to a situation of a chain of connecting orbits $u_1, \cdots, u_k$ in moduli spaces $\mathcal{M}(x_1, x_2) \times \mathcal{M}(x_2, x_3) \times \cdots \times \mathcal{M}(x_k, x_{k+1})$.

If we add such chains of connecting orbits as a boundary to the moduli space $\mathcal{M}(x, y)$, we may hope that the resulting moduli spaces are compact. For this, we should have that every sequence of connecting cylinders $(u_n)_{n \in \mathbf{N}}$ in $\mathcal{M}(x, y)$ should have a sub-sequence that converges uniformly to some chain $\tilde{u}_1, \cdots, \tilde{u}_k$. This is not in general true, however, because in certain cases, a sequence of connecting cylinders may develop a 'bubble'. We will now describe this phenomenon, and also see how we can rule it out in the case that is relevant for us.

For an almost complex structure $J$, we define a $J$-holomorphic sphere as a smooth map $v \colon S^2 \to M$ such that $\mathrm{d}v \circ \mathrm{i} = J \circ \mathrm{d}v$, where i is the complex structure obtained from $S^2 \cong \mathbf{C}P^1$. The energy $E(v)$ of $v$ is defined as $\int_{S^2} v^* \omega$ and is a positive quantity. It is a fact that, in the monotone case, $E(v)$ is bounded from below by $N/\tau$.

**Proposition 2.6.1** (convergence modulo bubbling)**.** *Let $(u_n)_{n \in \mathbf{N}}$ be a sequence in $\overline{\mathcal{M}}(x^-, x^+)$ with bounded energy. Then there exists a curve $u$ in some moduli space $\overline{\mathcal{M}}(x, y)$ such that, on compact sets away from finitely many points $z_1, \cdots, z_\ell \in \mathbf{R} \times S^1$, the sequence $(u_n)_{n \in \mathbf{N}}$ converges with derivatives to $u$. The energy of $u$ is bounded by*

$$E(u) \leq \limsup_{n \to \infty} E(u_n) - \ell N / \tau$$

□

One should think of the sequence as developing $J_t$-holomorphic spheres at $z = (s, t)$, which makes the energy bound intuitively plausible.

We also have, similarly to proposition 1.2.2, a

**Corollary 2.6.2.** *Let $(u_n)_{n\in\mathbf{N}}$ be a sequence in $\mathcal{M}(x^-,x^+)$ with bounded energy. Then there exists a chain of curves $\tilde{u}_1,\cdots,\tilde{u}_k$ and offsets $(s_{n,1}),\cdots,(s_{n,k})$ such that $u_n(s+s_{n,i},t)$ converges to $\tilde{u}_i$ modulo bubbling. The limit cylinders satisfy*

$$\sum_{j=1}^{k} E(\tilde{u}_j) \leq \limsup_{n\to\infty} E(u_n) - \ell N/\tau$$

*where $\ell$ is the total number of bubbles.* □

We will now prove that, in the monotone case, the moduli spaces $\mathcal{M}^1(x^-,x^+)$, the one-dimensional component(s) of $\mathcal{M}(x^-,x^+)$, and $\mathcal{M}^2(x^-,x^+)$ are compactified by adding broken flow lines as a boundary. This is then sufficient to define a boundary operator $d$ (since $\mathcal{M}^1(x^-,x^+)/\mathbf{R}$ will be finite) and to prove $d^2=0$ (by an argument analogous to the one in Morse homology). It all comes down to proving that, in the monotone case, there is not enough energy for bubbling to occur.

**Proposition 2.6.3.** *The space $\mathcal{M}^i(x^-,x^+)$ is compactified by broken flow lines for $i=1,2$.*

*Proof.* For any $u\in\mathcal{M}^i(x^-,x^+)$ we have, by (2.9),

$$\eta_H(x^-)-\eta_H(x^+)+2\tau E(u)=i$$

Then a sequence $u_n$ has bounded energy, so it has a sub-sequence (also $u_n$) converging to a limit chain $\tilde{u}_1,\cdots,\tilde{u}_k$ in moduli spaces $\mathcal{M}_j=\mathcal{M}(x_j,x_{j+1})\ni\tilde{u}_j$. Applying to same formula to a limit solution we obtain

$$\begin{aligned}\sum_{j=1}^{k}\dim_{\tilde{u}_j}\mathcal{M}_j &= \sum_{j=1}^{k}\left(\eta_H(x_j)-\eta_H(x_{j+1})+2\tau E(u_j)\right)\\ &= \eta_H(x^-)-\eta_H(x^+)+2\tau\sum_{j=1}^{k}E(u_j)\\ &\leq \eta_H(x^-)-\eta_H(x^+)+2\tau E(u)-2\tau\ell N/\tau\\ &= i-2\ell N\end{aligned}$$

For monotone $(M,\omega)$, $N$ is positive, and so are all the dimensions on the LHS. Then we see that $\ell=0$. In other words, $u$ converges to the chain $\tilde{u}_1,\cdots,\tilde{u}_k$ without any bubbling. □

## 2.7 THE FLOER HOMOLOGY COMPLEX

The definition of the Floer homology complex is somewhat more involved than in Morse theory, because only relative indices depending on homology classes are defined. To deal with this, we introduce extra variables keeping track of the chosen homology classes.

We assume that $H_2(W, \mathbf{Z})$ is torsion-free. Suppose that $A_1, \cdots, A_k$ is a $\mathbf{Z}$-basis for the image of $\pi_2(W)$ in $H_2(W, \mathbf{Z})$. We associate formal graded variables $z_1, \cdots, z_k$ to this basis of degree $-2c_1(A_i)$. We define the *Novikov ring* $\Lambda$ of $W$ as the graded ring of formal sums

$$\sum_{d_1,\cdots,d_k \in \mathbf{Z}} c_{d_1,\cdots,d_k} z^{d_1} \cdots z^{d_k}$$

satisfying the finiteness condition

$$\#\{(d_1, \cdots, d_k) \mid c_{d_1,\cdots,d_k} \neq 0, \omega(\sum_i d_i A_i) < c\} < \infty$$

for any $c \in \mathbf{R}$.

We associate graded formal variables $q_\gamma$ of degree $\mu_{\mathrm{CZ}}(\gamma, v_\gamma)$ (for some choice of spanning surface $v_\gamma$) to every periodic orbit $\gamma$, and we define the *Floer homology chain complex* $CF^\bullet(W)$ as the free graded module over the Novikov-ring generated by these variables. We define a boundary operator $d$ given by

$$d(q_{\gamma^-}) := \sum_{\{\gamma^+ | \overline{\mathcal{M}}(\gamma^-,\gamma^+)/\mathbf{R} \text{ is finite}\}} \sum_{u \in \overline{\mathcal{M}}(\gamma^-,\gamma^+)/\mathbf{R}} z^{[(-v_{\gamma^-})\#u\#v_{\gamma^+}]} \cdot q_{\gamma^+}$$

and extended $\Lambda$-linearly. Then we have

**Proposition 2.7.1.** *Suppose $(M, \omega)$ is monotone. Then the operator $d$ is a chain operator, i.e. $d$ is of degree $-1$ and $d^2 = 0$.*

*Proof.* The fact that $d$ is of degree $-1$ follows easily from the two observations that (1) $\overline{\mathcal{M}}(\gamma^-, \gamma^+)/\mathbf{R}$ is 0-dimensional iff the difference $\mu_{\mathrm{CZ}}(\gamma^+, v^-\#u) - \mu_{\mathrm{CZ}}(\gamma^-, v^-) = 1$; and (2) that by (2.7), the indices $\mu_{\mathrm{CZ}}(\gamma^+, v^-\#u)$ and $\mu_{\mathrm{CZ}}(\gamma^+, v^+)$ differ exactly by $c_1((-v_{\gamma^-})\#u\#v_{\gamma^+})$.

The fact that $d^2 = 0$ follows by exactly the same argument as in corollary 1.2.3, taking into account that the homology class of $u$ (relative to $\gamma^\pm$) does not change in a component of $\overline{\mathcal{M}}(\gamma^-, \gamma^+)$. □

We define $HF^\bullet(W)$ to be the homology of the chain complex $(CF^\bullet(W), d)$.

For suitable classes of manifolds, it can be proved that this homology is the same as singular homology with coefficients in $\Lambda$. Beyond the monotone case, the introduction of the Novikov ring allows one to bound the energy, as needed for the proof of 2.6.3, even in the case $\tau = 0$. We will not attempt to go into this, but instead refer the interested reader, once again, to [13].

# Part II

# SYMPLECTIC FIELD THEORY

# 3

# GEOMETRICAL SETUP

## 3.1 CONTACT STRUCTURES

A *contact manifold* is a pair $(V, \xi)$ with $V$ a real manifold and $\xi$ a sub-bundle of the tangent bundle $TV$ (so $\xi$ is a *distribution*) of co-dimension 1, satisfying the following: locally at every point, there is a 1-form $\alpha$ such that $\xi = \ker \alpha$ and the restriction of $\mathrm{d}\alpha$ to $\xi$ is non-degenerate. Again, this means (since $\mathrm{d}\alpha$ is antisymmetric) that fibres of $\xi$ have even dimension so $V$ must have odd dimension.

When $\xi$ is co-orientable, we can fix $\alpha$ globally. (However, for a given contact structure, this choice is not unique.) We will assume co-orientability, and a choice of $\alpha$ having been made, in what follows.

A contact manifold with a contact form is an example of a *stable Hamiltonian structure* in the sense of [1]. This is a triple $(V, \omega, \alpha)$ such that

1. $V$ is a $(2n-1)$-dimensional manifold;
2. $\omega$ is a closed 2-form that is maximally non-degenerate, i.e. $\ker \omega = \{v \in TV \mid \omega(v, \cdot) \equiv 0\}$ is one-dimensional;
3. $\mathrm{d}\alpha$ has $\ker \omega \subseteq \ker \mathrm{d}\alpha$;
4. $\alpha(v) \neq 0$ for $v \in \ker \omega - \{0\}$.

For a contact manifold we just take $\omega = \mathrm{d}\alpha$. Other examples of stable Hamiltonian structures are principal circle bundles $\pi \colon V \to M$ over symplectic manifolds $M$. In this case, we take $\pi^*\omega_M$ for $\omega$ and $\alpha$ can be any $S^1$-connection form.

We can associate to $(V, \omega, \alpha)$ a vector field $R$ on $V$, called the *Reeb vector field.* We choose $R$ in the one-dimensional distribution $\ker \omega$ and normalise by requiring $\alpha(R) = 1$. The flow of this vector field is called the *Reeb flow.* The Reeb flow preserves $\alpha$, since for the Lie derivative of $\alpha$ along $R$ we have by definition

$$\mathcal{L}_R \alpha \quad = \quad \iota_R \mathrm{d}\alpha + \mathrm{d}(\iota_R \alpha)$$

The first term is zero because $R \in \ker \omega \subseteq \ker d\alpha$ and the second term is zero because it is $d(\alpha(R)) = d(1)$.

Periodic flow lines of the Reeb flow are called *Reeb orbits*. We do not restrict to simple orbits; if $\gamma$ is a Reeb orbit, we write $\kappa_\gamma$ for its multiplicity.

For simplicity, we will write the exposition of symplectic field theory using the language of contact manifolds, but this can be generilised to stable Hamiltonian structures.

## 3.2 SYMPLECTIC COBORDISMS

We will now study *symplectic cobordisms*. The simplest example of these is the symplectization of a contact manifold $(V, \xi)$. It is defined as follows. Remember that $\xi$ is a sub-bundle of $TV$. Therefore, we can consider the quotient bundle $TV/\xi$ and its dual $(TV/\xi)^\vee$. A fixed contact form $\alpha$ defines a trivialisation $(TV/\xi)^\vee = V \times (\mathbf{R} \backslash \{0\})$. We pick the positive half $\mathbf{R}_{>0}$, give it the coordinate $e^t$, and call the symplectic manifold $(V \times \mathbf{R}_{>0}, d(e^t\alpha))$ the *symplectization* of $V$. It does not depend on $\alpha$ (but the factorisation $V \times \mathbf{R}_{>0}$ does).

By changing coordinates $e^t \mapsto t$ we can describe the symplectization as $V \times \mathbf{R}$, and we will do so in what follows. We call $V \times (-\infty, 0]$ the *negative cylindrical end* and $V \times [0, \infty)$ the *positive cylindrical end*. This allows us to define:

**Definition 3.2.1.** A symplectic manifold $(W, \omega)$ is called a *symplectic cobordism* if the following holds:

1. there is a compact contact manifold $V^-$ such that $W$ has a subset $E^-$ symplectomorphic to the negative cylindrical end of the symplectization of $V^-$;

2. there is a compact contact manifold $V^+$such that $W$ has subset $E^+$ symplectomorphic to the positive cylindrical end of the symplectization of $V^+$;

3. The subset that is the closure of $W \backslash (E^- \cup E^+)$ is compact.

In this case we write $W = \overrightarrow{V^-V^+}$. We see that the symplectization of a contact manifold $V$ is a cobordism $\overrightarrow{VV}$. We call such cobordisms *cylindrical*.

We will now define the operation of "splitting" a symplectic cobordism $W = \overrightarrow{V^-V^+}$ along a contact manifold $V \subseteq W$, obtaining two cobordisms $\overrightarrow{V^-V}$ and $\overrightarrow{VV^+}$. So let $V \subseteq W$ be a manifold of dimension $2n - 1$, which admits a contact form $\alpha|_V$ such that it is a restriction of some form $\alpha$ with $d\alpha = \omega$ locally near $V$. Assume that $W\backslash V$ has two connected components $W^-$ and $W^+$ with $V$ as their common boundary. Then we define the symplectic cobordisms

$$\begin{aligned} W^-_\infty &:= W^- \cup V \times [0,\infty) \\ W^+_\infty &:= V \times (-\infty,0] \cup W^+ \end{aligned}$$

with contact forms given by $\omega|_{W^-}$ together with $\mathrm{d}\big(\mathrm{e}^t\alpha\big)$, (which can be pasted smoothly), respectively $\omega|_{W^+}$ together with $\mathrm{d}\big(\mathrm{e}^t\alpha\big)$. The notation $W^\pm_\infty$ is explained by the following limiting process: define the symplectic manifolds

$$\begin{aligned} W^-_\tau &:= W^- \cup V \times [0,\tau) \\ W^+_\tau &:= V \times (\tau,0] \cup W^+ \\ W_\tau &:= W^- \cup V \times [-\tau,\tau] \cup W^+ \end{aligned} \tag{3.1}$$

(we should regard $W^\pm_\tau$ as subsets of $W_\tau$). Then $W_0 = W$ and the $W_\tau$s are deformations of $W$, splitting $W$ into the two cobordisms $W^\pm_\infty$ in the limit $\tau \to \infty$. We write $W = W^-_\infty \circledcirc W^+_\infty$, and also (abusing notation) $W = W^- \circledcirc W^+$.

## 3.3 STABLE CURVES IN A COMPACT SYMPLECTIC MANIFOLD

Let $(W,\omega)$ be symplectic of dimension $2n$. Let $J$ be a almost complex structure on $W$. Let $C$ be a connected Riemann surface and i the complex structure on its tangent space. Then a *$J$-holomorphic curve* is a smooth map $f\colon C \to W$ such that $\mathrm{d}f \circ \mathrm{i} = J \circ \mathrm{d}f$. If we mark $r$ distinct points $y_1,\cdots,y_r \in C$, then we call the tuple $(f,y_1,\cdots,y_r)$ a $J$-holomorphic curve with $r$ marked points. We say that two such curves with marked points, say $(f\colon C \to W, y_1,\cdots,y_r)$ and $(f'\colon C' \to W, y'_1,\cdots,y'_r)$, are equivalent if there is an isomorphism $\phi\colon C \to C'$ with $\phi(y_i) = y'_i$ and $f' \circ \phi = f$.

Suppose $W$ is compact and let $A \in H_2(W,\mathbf{Z})$. We write $\mathcal{M}^A_{g,r}(W,J)$ for the moduli space of $J$-holomorphic curves $(f\colon C \to W, y_1,\cdots,y_r)$ for which $C$ is smooth, compact and of genus $g$, for which $f_*([C]) = A$ (here $[C]$ is the fundamental cycle on $C$), identifying equivalent curves, and requiring the following *stability condition*:

> If $f$ is constant and if $g = 0$ (resp. 1), then $C$ has at least 3 (resp. 1) marked points.

**Example 3.3.1.** In the situation $W = \{\mathrm{pt}\}$ and $g = 0$, we need at least three marked points to satisfy the stability criterion. All tuples $(f,y_1,y_2,y_3)$, with $f$ the constant map $\mathbf{C}P^1 \to \{\mathrm{pt}\}$, are equivalent and we can ignore $A$ since $H_2(\{\mathrm{pt}\}) = 0$. So $\mathcal{M}_{0,3}(\{\mathrm{pt}\})$ is the singleton $\{(f,0,1,\infty)\}$. If we add a fourth marked point, we can add it anywhere on $\mathbf{C}P^1$ away from the first three points, so $\mathcal{M}_{0,4}(\{\mathrm{pt}\}) = \mathbf{C}P^1\backslash\{0,1,\infty\}$. In general, we can describe $\mathcal{M}_{0,r}(\{\mathrm{pt}\})$ as the quotient of $\big(\mathbf{C}P^1\big)^r\backslash\Delta$

by the simultaneous action of $\mathrm{Aut}(\mathbf{C}P^1)$ on the factors, where $\Delta$ is the closed subset $\{y_i = y_j \text{ for some } i \neq j\}$.

We are interested in compactifying $\mathcal{M}^A_{g,r}$. We do this by allowing curves $C \to W$ for which $C$ has simple nodes. We must still require stability, specifically:

> If $f\colon C \to W$ is constant *when restricted to an irreducible component* $S \subseteq C$, and $g(S) = 0$ (resp. 1), then $S$ has least 3 (resp. 1) special points, i.e. marked points and/or nodes.

If $C$ has simple nodes, we should specify more precisely that $g$ refers to the *arithmetic* genus. When a sequence of curves converges to a curve with a node, this is sometimes referred to as *bubbling*.

The resulting moduli space is denoted $\overline{\mathcal{M}}^A_{g,r}(W, J)$.

*Remark* 3.3.2. The moduli spaces we are considering are, in general, quite complicated objects. They are certainly not manifolds, and they often even fail to be orbifolds. According to [3], they are "branched-labelled orbifold with boundaries and corners" and even then only "after choosing abstract perturbations using polyfolds". Alternatively, algebraic geometers have introduced the notion of "stack".

We will not go into definitions and properties of these structures. We will also not need this; we will only use the fact that a moduli space can be given a structure 'making it work' as if it were a manifold. Here, 'making it work' means that we can talk about its dimension, homology, cohomology as if it were manifold. In particular, we will use Stokes' theorem and Poincaré duality. This is what we will mean by saying that a certain set is a moduli space.

We have the following theorem about it

**Theorem 3.3.3.** *The compactified moduli space* $\overline{\mathcal{M}}^A_{g,r}(W, J)$ *has dimension*

$$(n-3)(2-2g) + 2c_1(A) + 2r\,.$$

*Here* $c_1$ *the first Chern class of the almost complex bundle* $(TW, J)$. □

**Example 3.3.4.** Let us compactify $\mathcal{M}_{0,4}(\{\mathrm{pt}\}) \cong \mathbf{C}P^1 \backslash \{0, 1, \infty\}$. Compactifying adds the nodal curves with two irreducible components with two marked points on either component. So

$$\overline{\mathcal{M}}_{0,4} = \mathbf{C}P^1 \backslash \{y_1, y_2, y_3\} \bigcup \{\text{three nodal curves}\}$$

We describe what has just happened as follows: Given a curve $C$ in the moduli space $\overline{\mathcal{M}}_{0,3}$, we get curves in $\overline{\mathcal{M}}_{0,4}$ by letting a new point $y_4$ run over the curve. When the new marked point $y_4$ hits an already marked point $y_i$, a new component

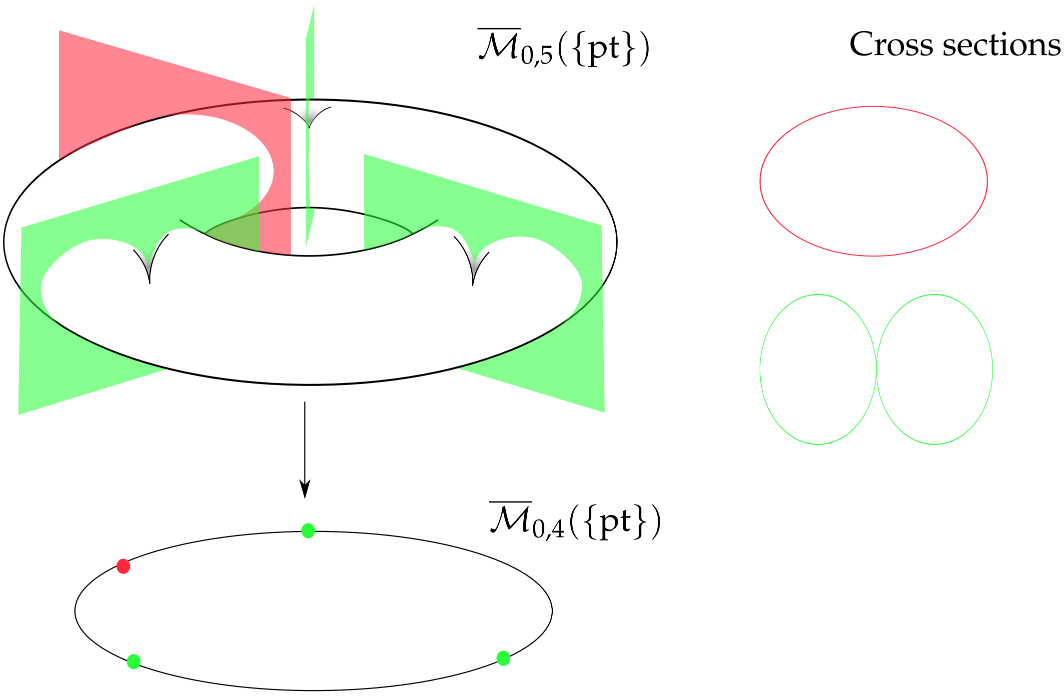



Figure 3.1: A half-dimensional drawing of the moduli space $\overline{\mathcal{M}}_{0,5}$ as a fibre bundle over $\overline{\mathcal{M}}_{0,4}$. The fibre over a curve $C \in \overline{\mathcal{M}}_{0,4}$ is isomorphic to $C$ itself. Depicted in green are cross-sections corresponding to nodal curves; depicted in red is a smooth curve.

isomorphic to $\mathbf{C}P^1$ "pops up"; we get a node where $y_i$ was; and the three points $y_i, y_4$ and the node ensure the stability of the new component.

In the same way, we can try to describe $\overline{\mathcal{M}}_{0,5}$. For every curve $C \in \overline{\mathcal{M}}_{0,4}$, we obtain a subset of curves in $\overline{\mathcal{M}}_{0,5}$ that can be identified with $C$ in the following way: When the new point $y_5$ hits a marked point, a new component pops up just as before. When the new point $y_5$ hits a node between two components, we insert a new component at the node. This new component has two nodes attaching to to the old components. These nodal points ensure, together with $y_5$, stability.

We write $\pi\colon \overline{\mathcal{M}}_{0,r+1} \to \overline{\mathcal{M}}_{0,r}$ for the map that forgets the last marked point, removing any components that thereby become unstable. In the case $W = \{\mathrm{pt}\}$, we see, in analogy to the previous example, that the fibre of $\pi$ above a curve $C \in \overline{\mathcal{M}}_{0,r}$ is isomorphic to $C$ itself. In other words, we can interpret $\overline{\mathcal{M}}_{0,r+1}$ as the *universal curve* over $\overline{\mathcal{M}}_{0,r}$.

## 3.4 $J$-HOLOMORPHIC CURVES IN COBORDISMS

Now consider the situation where $W$ is a cobordism. In this case, we will restrict our choice of the almost complex structure $J$ by specifying that in the cylindrical ends $(V^\pm \times \mathbf{R}_\pm, \xi^\pm, \mathrm{d}(\mathrm{e}^t\alpha))$, it is translation invariant along $\mathbf{R}_\pm$ (if $W$ is not cylindrical itself, this needs only hold for large enough $t$); that it preserves the contact structure $\xi^\pm$ on $V^\pm$; and that it sends the field $\frac{\partial}{\partial t}$ to the Reeb vector field along $V^\pm$.

ASYMPTOTIC BEHAVIOUR We fix a number of punctures $x_1, \cdots, x_s$ (distinct from the marked points) on the compact connected Riemann surface $C$. We will allow $C \to W$ to "run off to $\pm\infty$" (in the cylindrical ends of $W$) in a neighbourhood of each of these $x_i$. If we regard a punctured neighbourhood of $x_i$ as a cylinder, then we will want it its end at infinity to map to a Reeb orbit in $V^\pm$. We will now make this precise and introduce the necessities for stability.

Let $x$ be a puncture in a stable curve, and $r\mathrm{e}^{\mathrm{i}\phi}$ holomorphic coordinates on a neighbourhood $U$ with $x$ at $r = 0$. Then the *real orientable blow-up at $x$* is the surface

$$\{(\theta, r\mathrm{e}^{\mathrm{i}\phi}) \in S^1 \times U \mid r = 0 \text{ or } \theta = \phi\}.$$

Away from $r = 0$, it is diffeomorphic to $U \backslash \{x\}$, so we can embed $C$ into its real orientable blowup, and this embedding does not depend on the choice of holomorphic coordinates. An *asymptotic ray at $x$* is a choice of a point $(\theta_0, 0)$ on the blow-up at $x$.

Let $\gamma^-$ be a Reeb orbit on the negative cylindrical end $V^-$. We endow it with a preferred starting point $\gamma^-(0)$, which we will call *asymptotic marker*. Then we say that a stable curve is *asymptotically cylindrical* at a puncture $x^-$ over $\gamma^-$ if:

1. The first coordinate $f_V$ of $f\colon U \to V \times \mathbf{R}$ extends to $\bar{f}_V$ on the real orientable blow-up at $x^-$;

2. In the negative cylindrical end $V \times (-\infty, 0]$, the second coordinate $f_{\mathbf{R}}$ satisfies $f_{\mathbf{R}}(re^{i\phi}) \to -\infty$ as $r \to 0$;

3. $\bar{f}_V(\theta_0 + \theta, 0)$ parametrises the Reeb orbit $\gamma^-(\theta)$ with the preferred starting point at $\theta = 0$.

At a Reeb orbit $\gamma^+$ in the positive cylindrical end $V^+$, we require:

1. There is an extension $\bar{f}_V$ to the real orientable blow-up at $x^+$;

2. In the positive cylindrical end $V \times [0, \infty)$, we have $f_{\mathbf{R}}(re^{i\phi}) \to \infty$ as $r \to 0$;

3. and $\bar{f}_{\mathbf{R}}(\theta_0 - \theta, 0)$ (note the change of orientation[1]) parametrises the Reeb orbit $\gamma^+(\theta)$ with the preferred starting point at $\theta = 0$.

Homology choices Now that we allow punctures in $C$, it makes no sense to talk about the homology class $[C]$ or its push-forward $f_*([C])$. Still, we want to distinguish curves that assume "homologically different" images in $W$. To do so, we make some homology choices at the cylindrical ends. We write $\Gamma^\pm$ for the sets of positive and negative Reeb orbits over which $f$ is asymptotically cylindrical.

Let us write $\overline{W} = V^- \sqcup W \sqcup V^+$; in other words, we extend the cylindrical ends $V^\pm \times \mathbf{R}_\pm$ to $V^\pm \times (\mathbf{R}_\pm \cup \{\pm\infty\})$. We also write $\overline{C}$ for the real orientable blow-up of $C$ at each of the punctures. Then the asymptotically cylindrical curves $f$ extend to $\overline{C}$ and $f_*([\overline{C}])$ gives a relative homology class in $H_2(\overline{W}, \Gamma^- \cup \Gamma^+)$.

We could, for all sets of Reeb orbits $\Gamma^\pm$, fix a cycle $A \in C_2(\overline{W})$ with $\partial A = \sum_{\gamma^+ \in \Gamma^+} [\gamma^+] - \sum_{\gamma^- \in \Gamma^-} [\gamma^-]$ and then to $(f, C)$ associate the homology class of the cycle $f_*([\overline{C}]) - A$. However, we want these homology classes to be additive in decompositions $W = W \circledcirc B$ or $W = B \circledcirc W$ (with $B$ cylindrical), so we cannot choose $A$ too freely; the part that is to cancel should only depend on one of $\Gamma^-, \Gamma^+$. The following construction takes care of this.

For a contact manifold $V$, we fix a basis $a_1, \cdots, a_k$ for $H_1(V)$, realised by paths $\delta_1, \cdots, \delta_k \in \Delta_1(V)$. For every Reeb orbit $\gamma \in V$ with homology class $\sum c_i a_i$ (we assume that $H_1(V)$ is torsion-free, so this is uniquely defined) we fix a cycle $A_\gamma \in C_2(V)$ with $\partial(A_\gamma) = \sum c_{\gamma,i} \delta_i - \gamma$.

1 I have not found this change of orientation mentioned explicitly in the literature. It is necessary in the following simple example: we can identify the symplectization of $S^1$ with $\mathbf{C}^*$. Then we want $\mathrm{id}\colon \mathbf{C}^* \to \mathbf{C}^*$ to be asymptotically cylindrical. Around the puncture at $\infty$, holomorphic coordinates $re^{i\phi}$ induce an orientation on $S^1$ opposite to the one induced by coordinates $re^{i\phi}$ around the puncture at 0. We should flip it back.

Now, fix a basis $b_1, \cdots, b_\ell$ for $H_1(W)$ realised by paths $\eta_1, \cdots, \eta_\ell \in \Delta_1(\overline{W})$. For the chosen $\delta_i^\pm \in C_1(V^\pm)$, fix cycles $B_i^\pm \in C_2(\overline{W})$ such that $\partial(B_i^\pm) = \sum c_j' \eta_j - \delta_i^\pm$. Then we have that

$$f_*([\overline{C}]) - \sum_{\gamma \in \Gamma^-} \left(A_\gamma + \sum c_{\gamma,i} B_i^-\right) + \sum_{\gamma \in \Gamma^+} \left(A_\gamma + \sum c_{\gamma,i} B_i^+\right)$$

is a closed cycle, giving rise to a homology class $A \in H_2(\overline{W})$. Choosing, in the cylindrical ends, any homeomorphism $\mathbf{R}_\pm \cup \{\pm\infty\} \to [0, \pm K]$ (for large $K$) gives a well-defined homology class $A \in H_2(W)$. This will be the homology class associated to the curve $f$.

CONLEY-ZEHNDER INDEX AND BAD ORBITS Similarly to Floer homology, we will need to define Conley-Zehnder indices for the Reeb orbits in order to calculate dimensions of moduli space of curves. This definition is almost the same as the definition given in section , but is slightly more complicated as the Reeb orbit may be homologically nontrivial. We refer to [2, § 1.2] for the definition.

In symplectic field theory, we must ignore certain multiple covers of Reeb orbits for technical reasons. We call a $\kappa$-fold cover $\gamma^\kappa$ of an orbit $\gamma$ *bad* if $\mu_{CZ}(\gamma^\kappa) \neq \mu_{CZ}(\gamma) \pmod 2$. We call it *good* otherwise. We will, without further mentioning this, only consider good Reeb orbits.

MODULI SPACE AND COMPACTIFICATION

**Definition 3.4.1.** Let $\Gamma^-$ and $\Gamma^+$ be ordered tuples of (good) Reeb orbits in $V^-$ and $V^+$ respectively. We write $s^\pm = \#\Gamma^\pm$. Then we define the moduli space

$$\mathcal{M}_{g,r}^A(\Gamma^-, \Gamma^+; W, J)$$

as the moduli space of $J$-holomorphic curves with $s^\pm$ negative (positive) punctures, asymptotically cylindrical above the orbits in $\Gamma^\pm$, with associated homology class $A$, where we identify two curves if there is an isomorphism between the two $C$'s that respects (the ordered sets of) marked points and punctures, and the asymptotic markers at the punctures.

The compactification of these moduli spaces will be a bit more involved. In addition to the "bubbling" phenomenon from the compact case, we now also have the following phenomenon: In a sequence of curves, parts of the curve may run off to $\pm\infty$ in the cylindrical ends. In this case, a natural limit may not be a single curve $f: C \to W$, but instead a chain of curves $f_1, \cdots, f_k$ in a decomposition $W = W_1 \circledcirc \cdots \circledcirc W_k$.

**Example 3.4.2.** As a sketchy example, consider the following. Suppose we have a sequence of curves $f_\tau \colon C \to W$, where $W = V \times \mathbf{R}$ is cylindrical and $C$ of genus 0, with two marked points. Suppose that, as $\tau \to \infty$, the two marked points are mapped $2\tau$ apart (in the $\mathbf{R}$-direction).

We can write $W = W_\infty^- \circledcirc W_\infty^+$ where both $W_\infty^\pm$ are symplectomorphic to $V \times \mathbf{R}$. Referring back to the notation in equation (3.1), we see that $W_\tau$ is isomorphic to $W$ for every $\tau$. We can therefore regard the $f_\tau$ as maps $f_\tau \colon C \to W_\tau$. In our example case, we will assume that (we can choose the isomorphisms $W_\tau \cong W$ such that) we have two open sets $U_\tau^\pm \subseteq C$ with

$$f_\tau^{-1}(W_\tau^\pm) = U_\tau^\pm$$

where both $f_\tau|_{U_\tau^+}$ and $f_\tau|_{U_\tau^-}$ converge to curves $f^\pm \colon U^\pm \to W_\infty^\pm$ . It is a fact that these open sets $U^\pm$ will be isomorphic to punctured compact Riemann surfaces. So we obtain a chain of two $J$-holomorphic curves, both with one marked point, as the limit of the family. It is not difficult to see (on an informal level) that the Reeb orbits of the positive punctures of $f^+$ must agree with the Reeb orbits of the negative punctures of $f^-$.

It is clear in this example that the curves $f^\pm$ are only defined up to translation along $\mathbf{R}$. For a proper definition of convergence of a sequence of curves to a chain, we refer to [2]. For our purposes, we need the following theorem that describes the compactification of $\mathcal{M}_{g,r}^A(\Gamma^-, \Gamma^+)$ that it entails:

**Theorem 3.4.3.** *Let $W = \overrightarrow{V^- V^+}$ be a symplectic cobordism. Let $f^k$ be any sequence of $J$-holomorphic curves in $\mathcal{M}_{g,r}^A(\Gamma^-, \Gamma^+)$.* Then there exists a split chain

$$W = A_1 \circledcirc \cdots \circledcirc A_a \circledcirc W \circledcirc B_1 \circledcirc \cdots \circledcirc B_b$$

where all $A_i$ and $B_i$ are cylindrical, and a chain of possibly nodal, *possibly disconnected* curves $f_1, \cdots, f_{a+b+1}$ to which $f_k$ converges as $k \to \infty$. The curves in the cylindrical parts are defined only up to translation.

*Proof.* This [2], theorem 1.6.2. □

We will define $\overline{\mathcal{M}}_{g,r}^A(\Gamma^-, \Gamma^+)$ as the set $\mathcal{M}_{g,r}^A(\Gamma^-, \Gamma^+)$ together with all limits of the above form. This indeed recovers all suitable combinations of curves in moduli spaces $\overline{\mathcal{M}}(A_1) \times \cdots \times \overline{\mathcal{M}}(B_b)$; this can be proved along the same lines as 'gluing' in Floer homology.

## 3.5 DIMENSIONS OF THE MODULI SPACE

In [2], the following conjecture is put forward:

**Conjecture 3.5.1.** *For a generic choice of $J$, the compactified moduli space $\overline{\mathcal{M}}^A_{g,r}(\Gamma^-,\Gamma^+;W,J)$ (see our convention about the structure of moduli spaces in remark 3.3.2) has dimension*

$$\sum_{\gamma\in\Gamma^+}\mu_{CZ}(\gamma^+)-\sum_{\gamma\in\Gamma^-}\mu_{CZ}(\gamma^-)+(n-3)(2-2g-s^--s^+)+2c_1(A)+2r\,, \quad (3.2)$$

*with $c_1$ the first Chern class of the almost complex bundle $(TW,J)$.*

In all that follows, we will work with the moduli space and with this formula as if this were a theorem. In particular, we will interpret this as saying that we may apply Stokes' theorem and Poincaré duality; see our remark 3.3.2. Note that, because the Euler characteristic $2-2g$ is additive on a disjoint union of curves, and because so are all other components of the formula, we can apply it to moduli spaces of *possibly disconnected* curves, as well.

When applying it both to a moduli space and to the description of its boundary given in theorem 3.4.3, we can calculate a co-dimension 1 stratum of the boundary of $\overline{\mathcal{M}}^A_{g,r}(\Gamma^-,\Gamma^+)$. This is easily done when using the following facts:

1. The expression $2-2g-s^--s^+$ is additive when pasting curves at punctures.
2. Writing $\tilde{\mathcal{M}}$ for a moduli space that allows disconnected curves, we have that

$$\dim\tilde{\mathcal{M}}/\mathbf{R}=\dim\tilde{\mathcal{M}}-1 \quad (3.3)$$

**Proposition 3.5.2.** *Under these conditions, a top dimension, dense open subset of the boundary $\partial\overline{\mathcal{M}}^A_{g,r}(\Gamma^-,\Gamma^+)$ is given by chains of* possibly disconnected *curves of length two. More precisely: this dense open subset is given by chains of curves in splittings*

$$\begin{aligned} W &= W\circledcirc\left(V^+\times\mathbf{R}\right)\\ \textit{or}\qquad W &= \left(V^-\times\mathbf{R}\right)\circledcirc W\end{aligned}$$

*where as usual, the curve in the cylindrical part is only defined up to translation.*[2]

*In terms of the moduli spaces, we have that (writing $\tilde{\mathcal{M}}$ for the moduli space of possibly disconnected curves)*

$$\tilde{\mathcal{M}}^{A_1}_{r_1,g_1}(\Gamma^-,B^+;W)\times\tilde{\mathcal{M}}^{A_2}_{g_2,r_2}(B^-,\Gamma^+;V^+\times\mathbf{R})/\mathbf{R}$$

2 In [2], this boundary stratum is described by additionally requiring that the curve in the cylindrical part has all but one connected component equal to a trivial cylinder without marked points. As far as I can tell, this contradicts (3.3). However, the most important result proposition 4.2.1 continues to hold, because in that case, only the curves satisfying this additional requirement contribute.

*and*

$$\tilde{\mathcal{M}}^{A_2}_{g_2,r_2}(\Gamma^-, B^+; V^- \times \mathbf{R})/\mathbf{R} \times \tilde{\mathcal{M}}^{A_1}_{r_1,g_1}(B^-, \Gamma^+; W)$$

*($A_1 + A_2 = A$, $r_1 + r_2 = r$, $g_1 + g_2 + \#\Gamma - 1 = g$, $B^- \cap B^+ \neq \emptyset$) are multiple covers of the appropriate stratum of the boundary; the multiplicity is $\kappa_{B^+ \cap B^-}$.*

In the case of a cylindrical cobordism $W = V \times \mathbf{R}$, we obtain:

**Proposition 3.5.3.** *A top dimension, dense open subset of the boundary $\partial\overline{\mathcal{M}}^A_{g,r}(\Gamma^-, \Gamma^+; W)/\mathbf{R}$ is given by chains of (possibly disconnected) curves of length two, where in both parts, at most one connected component is not a trivial cylinder. More precisely: this dense open subset is given by chains of curves in a splitting*

$$W \quad = \quad W \circledcirc W$$

*where both curves may have only one connected component that is not a trivial cylinder without marked points.*

*In terms of the moduli spaces, we have that*

$$\mathcal{M}^{A_1}_{r_1,g_1}(\Gamma^-, B^+; W)/\mathbf{R} \times \mathcal{M}^{A_2}_{g_2,r_2}(B^-, \Gamma^+; W)/\mathbf{R}$$

*($A_1 + A_2 = A$, $r_1 + r_2 = r$, $g_1 + g_2 + \#\Gamma - 1 = g$, $B^- \cap B^+ \neq \emptyset$) is a multiple cover of an appropriate stratum of the boundary; the multiplicity is $\kappa_{B^+ \cap B^-}$.*

Again, the trivial cylinder components are implied by the non-matching Reeb orbits.

# 4

# ALGEBRAIC DESCRIPTION

## 4.1 PRELIMINARY REMARKS

The algebraic objects we will be considering will be algebras over **C**, and some will have additional structure. Let us review the definitions.

### GRADED ALGEBRAS

**Definition 4.1.1.** An *algebra* (over **C**) is a unitary, possibly non-commutative ring $A$, together with a multiplicative **C**-action that is distributive over both the ring addition and over addition in **C**, and that commutes with ring multiplication. A (**Z**-)*graded algebra* is an algebra $A$ that can be written as a direct sum

$$A = \bigoplus_{d=-\infty}^{\infty} A_d$$

such that if $a \in A_d$ and $b \in A_e$, then $ab \in A_{d+e}$. The elements $a$ of $A_d$ are said to be *homogeneous* of degree $d$; we write $\deg a = |a| = d$. A pair of homogeneous elements $a, b$ in a graded algebra are said to *super-commute* if

$$ab = (-1)^{|a|\cdot|b|} ba \,.$$

An algebra $A$ is *super-commutative* if all pairs of homogeneous elements super-commute.

**Example 4.1.2.** Rings of polynomials (in any number of variables) are examples of graded algebras, with deg given by the total degree of a monomial. The exterior algebra $\bigwedge^\bullet(V)$ on a vector space $V$ is an example of a super-commutative algebra.

**Definition 4.1.3.** For two homogeneous elements $a, b$ in a graded algebra $A$, we define the (graded) *commutator* $[a, b]$ as

$$[a, b] = ab - (-1)^{|a|\cdot|b|} ba$$

We extend it to all elements in $A$ bilinearly.

Note that $[a,b]=0$ if and only if $a$ and $b$ super-commute. We will often want to write a polynomial in terms of monomials with a prescribed ordering for the variables. To do so, we will use the identity

$$pq = (-1)^{|p||q|} qp + [p,q] \,.$$

Applying it inductively, we obtain the following helpful

**Lemma 4.1.4.** *Let $p_1, \cdots, p_n, q_1, \cdots, q_m \in A$. Suppose all commutators $[p_i, q_j]$ are in the centre of $A$. Then by applying the above inductively, we obtain (writing $p_1 \cdots p_n =: p^B$ and $q_1 \cdots q_m =: q^C$)*

$$p^B q^C \quad = \sum_{\{(b_i,c_i)\}_i} \pm q^{C'} p^{B'} \prod_i [p_{b_i}, q_{c_i}]$$

*where the sum runs over all matchings*

$$\{(b_1,c_1), \cdots, (b_k,c_k)\} \subseteq B \times C = \{1 \cdots n\} \times \{1 \cdots m\}$$

*(of any size from $0$ to $\min(n,m)$); where we write $C' = C \setminus \cup_i \{c_i\}$ and $B' = B \setminus \cup_i \{b_i\}$; and where the sign is given by*

$$(-1)^{|p^{B'}||q^{C'}|} \prod_i (-1)^{|p_{b_i}||q_1 \cdots q_{c_i - 1}|}$$

□

BRACKETS

**Definition 4.1.5.** Given a graded algebra $A$, a *(graded) Poisson bracket* on $A$ is a bilinear pairing

$$\{\cdot,\cdot\} \colon A \times A \to A$$

that is anti-symmetric in the super-commutative sense, i.e. for homogeneous elements we have the identity

$$\{a,b\} = -(-1)^{|a|\cdot|b|} \{b,a\}$$

and that satisfies both the (graded) Jacobi identity

$$\{a,\{b,c\}\} + (-1)^{|a|\cdot|b|\cdot|c|} \{c,\{a,b\}\} + (-1)^{|a|\cdot|b|\cdot|c|} \{b,\{c,a\}\} = 0\,.$$

and the (graded) Leibniz identity

$$\{a,bc\} = \{a,b\}c + (-1)^{|a|\cdot|b|} b\{a,c\}$$

PRESENTATION OF NON-COMMUTATIVE ASSOCIATIVE ALGEBRAS We will often present our algebras as follows. For a countable set of variables $\{x_i\}$ we can define the algebra

$$\tilde{A} := \mathbf{C}[\{x_i\}] \quad \text{(non-commutative)}$$

by which we mean that elements of $\tilde{A}$ are finite[1] linear combinations of finite words in the $x_i$. If we specify commutation relations $[x_i, x_j] = h_{ij}$ (where $\deg h_{ij} = \deg x_i x_j$), then we obtain the algebra

$$A := \tilde{A} / \left( \cup_{i,j} [x_i, x_j] - h_{ij} \right) \tag{4.1}$$

We just describe this algebra $A$ as "the algebra of polynomials in the $x_i$ with the given commutation relations", without referring to $\tilde{A}$.

If we want to define a linear operator $d\colon A \to A$, we can first define $\tilde{d}\colon \tilde{A} \to \tilde{A}$; then it is sufficient to verify that $[x_i, x_j] - h_{ij} \in \ker \tilde{d}$ for the operator $d$ to be unambiguous.

PARTIAL DERIVATIVES

**Definition 4.1.6.** Let $\tilde{A} \to A$ be as in (4.1). Suppose $q \in \tilde{A}$ is a formal variable. Then we define the operator $\frac{\tilde{\partial}}{\partial q}\colon \tilde{A} \to \tilde{A}$ by requiring:

1. $\frac{\tilde{\partial}}{\partial q}$ is $\mathbf{C}$-linear;
2. $\frac{\tilde{\partial}}{\partial q} q = 1$ ;
3. For all other formal variables $a \in \tilde{A}$, $\frac{\tilde{\partial}}{\partial q} a = 0$ ;
4. For homogeneous elements $a, b$ we have the graded Leibniz identity
$$\frac{\tilde{\partial}(ab)}{\partial q} = \frac{\tilde{\partial} a}{\partial q} b + (-1)^{\deg a \deg q} a \frac{\tilde{\partial} b}{\partial q} .$$

Note that $\frac{\tilde{\partial}}{\partial q}[q, a] = 0$ for all $a \in \tilde{A}$. Suppose that in $A$ we have $[q, a] = h_a$, and that all $h_a$ are in $\ker \frac{\tilde{\partial}}{\partial q}$. Then $\frac{\tilde{\partial}}{\partial q}$ descends to an operator $\frac{\partial}{\partial q}\colon A \to A$.

*Remark* 4.1.7. It is important to realise how some of these graded operations and identities notably differ from their (perhaps more familiar) commutative versions. For example, for odd variables $q$, the vanishing of $[q, q]$ is not tautological: If $[q, q] = 0$, this means that $2q^2 = 0$. In particular, in a super-commutative algebra (where we do have $[q, q] = 0$ by definition), all odd elements square to zero. This is compatible with the graded Leibniz identity, which gives $\frac{\partial}{\partial q} q^2 = 0$ in this case.

1 We can also allow infinite linear combinations satisfying some finiteness condition.

The following construction shows a situation in which there is a clear relation between commutators and Poisson brackets.

**Example 4.1.8.** Let $A$ be a graded algebra containing an element $\hbar$ that is in the centre of $A$. Suppose that all commutators $[a, b]$ are in $\hbar A$. Then the algebra $P := A/\hbar A$ is super-commutative. We define a Poisson bracket on $P$ by the formula (for $\bar{a}, \bar{b} = a, b \mod \hbar A$)

$$\{\bar{a}, \bar{b}\} = \left(\frac{i}{\hbar}[a, b]\right)|_{\hbar \mapsto 0}$$

Then $P$ is called the *semi-classical approximation* to $A$.

## 4.2 HAMILTONIAN ASSOCIATED TO A CONTACT MANIFOLD

Given a contact manifold and contact form $(V, \alpha)$ of dimension $2n-1$, and its symplectization $(W, \omega) = (V \times \mathbf{R}, \mathrm{d}(\mathrm{e}^t\alpha))$, we collect some of the data about its moduli spaces of stable curves in algebraic structures. We associate formal variables to geometric objects in the following way:

1. We fix a basis $(\Theta_1, \cdots, \Theta_k)$ of closed forms for $H^\bullet(V)$ and introduce associated formal variables $t^1, \cdots, t^k$.
2. We introduce formal variables $p_\gamma$ and $q_\gamma$ for every good Reeb orbit $\gamma$ in $V$. The $p$ variables should be thought of as being associated to the positive, and the $q$ variables as being associated to the negative cylindrical end of $V \times \mathbf{R}$.
3. We fix a basis $(A^1, \cdots, A^j)$ for $H_2(W, \mathbf{Z})$ and to $A = d_i A^i \in H_2(W, \mathbf{Z})$ we associate an expression $z_1^{d_1} \cdots z_j^{d_j}$ (we will also write $z^A$ for short).

We also introduce maps $\mathrm{ev}_i \colon \overline{\mathcal{M}}^A_{g,r}(\Gamma^-, \Gamma^+)/\mathbf{R} \to V$, sending a stable curve $C$ to the image in $V$ of its $i$th marked point. Pasting these together, we obtain

$$\mathrm{ev} \colon \overline{\mathcal{M}}^A_{g,r}(\Gamma^-, \Gamma^+)/\mathbf{R} \to \overbrace{V \times \cdots \times V}^{r \text{ times}}.$$

We will use this map to pull forms on $V$ back to the moduli space.

The dimension formula (3.2) reads (slightly reordered)

$$\dim \overline{\mathcal{M}}^A_{r,g}(\Gamma^-, \Gamma^+)/\mathbf{R} = \sum_{\gamma^+ \in \Gamma^+} \big(\mu_{\mathrm{CZ}}(\gamma^+) - (n-3)\big) + \sum_{\gamma^- \in \Gamma^-} \big(-\mu_{\mathrm{CZ}}(\gamma^-) - (n-3)\big) \\ - (n-3)(2-2g) + 2c_1(A) + 2r - 1$$

We collect the different summands of this formula in a grading of the variables introduced above. We define:

$$\begin{aligned} \deg p_\gamma &:= (n-3) - \mu_{CZ}(\gamma) \\ \deg q_\gamma &:= (n-3) + \mu_{CZ}(\gamma) \\ \deg t^i &:= \deg \Theta_i - 2 \\ \deg z_i &:= -2c_1(A^i) \\ \deg \hbar &:= 2(n-3)\,, \end{aligned} \tag{4.2}$$

and we define the *Weyl algebra* $\mathfrak{W} = \mathfrak{W}(V, \alpha)$ as the graded algebra of power series in $\hbar$ with coefficients that are power series in the $t^i$ and $z_i$ and polynomials in the $p_\gamma, q_\gamma$; all graded commutators are zero, except for the $p$ and $q$ variables of the same Reeb orbit $\gamma$, for which we let

$$[p_\gamma, q_\gamma] = \kappa_\gamma \hbar \tag{4.3}$$

(remember that $\kappa_\gamma$ is the multiplicity of the orbit). We will write monomials in $\mathfrak{W}$ concisely as

$$t^{i_1} \cdots t^{i_r} q_{\gamma_1} \cdots q_{\gamma_{s^-}} p_{\gamma_1} \cdots p_{\gamma_{s^+}} z^A = t^I q^{\Gamma^-} p^{\Gamma^+} z^A$$

and also

$$\kappa_{\gamma_1} \cdots \kappa_{\gamma_s} = \kappa_\Gamma$$

The grading of the variables is chosen such that if we have indices $I = (i_1, \cdots, i_k)$ for which $\Theta_{i_1} \otimes \cdots \otimes \Theta_{i_k}$ is an $r$-form, then

$$\deg \Theta_{i_1} \otimes \cdots \otimes \Theta_{i_k} - \dim \mathcal{M}^A_{r,g}(\Gamma^-, \Gamma^+)/\mathbf{R} = 1 + \deg \frac{1}{\hbar} t^I q^{\Gamma^-} p^{\Gamma^+} z^A\,.$$

This mean that if we choose $\Gamma^\pm$, $I$ and $A$ such that $\deg \frac{1}{\hbar} q^{\Gamma^-} p^{\Gamma^+} t^I z^A = -1$, the form $\Theta_{i_1} \otimes \cdots \otimes \Theta_{i_k}$ on $V$ has the same degree as the dimension of $\overline{\mathcal{M}}^A_{r,g}/\mathbf{R}$. So the following definition gives a homogeneous element $\mathbf{H} \in \frac{1}{\hbar}\mathfrak{W}$ of degree $-1$:

$$\mathbf{H} := \frac{1}{\hbar} \sum_g \sum_{A,r,s^\pm} \sum_{I,\Gamma^\pm} \frac{1}{r!s^-!s^+!} \frac{1}{\kappa_{\Gamma^-}\kappa_{\Gamma^+}} \int_{\overline{\mathcal{M}}^A_{g,r}(\Gamma^-,\Gamma^+)/\mathbf{R}} \mathrm{ev}^*(\Theta_I)\, t^I q^{\Gamma^-} p^{\Gamma^+} z^A \hbar^g$$

We refer to $\mathbf{H}$ as the *(SFT -) Hamiltonian* for $V$. It is the subject of the following foundational proposition.

**Proposition 4.2.1.** *The Hamiltonian* $\mathbf{H}$ *satisfies*[2]

$$[\mathbf{H}, \mathbf{H}] = 0$$

2 This depends on the $\Theta_i$'s being assumed closed. The equation is often known by the name "master equation".

*Proof.* We have $[\mathbf{H},\mathbf{H}] = 2\mathbf{H}\cdot\mathbf{H}$ so we can equivalently prove $\mathbf{H}\cdot\mathbf{H} = 0$. This product has summands (omitting the coefficient)

$$t^I q^A p^B \hbar^{g_1-1} t^J q^C p^D \hbar^{g_2-1}\,.$$

We put the variables in normal order:

$$\begin{aligned} t^I q^A p^B h^{g_1-1} t^J q^C p^D h^{g_2-1} &= \pm t^{I\cup J} q^A p^B q^C p^D h^{g_1+g_2-2} \\ &= \pm \sum_{\substack{\text{matchings in}\\ B\times C}} \pm t^{I\cup J} q^{A\cup C'} p^{B'\cup D} \prod_i [p_{b_i}, q_{c_i}] \hbar^{g_1+g_2-2} \end{aligned}$$

(where we use the notation introduced in 4.1.4). A summand is only nonzero if all commutators $[p_{b_i}, q_{c_i}]$ are, which happens exactly if all pairs of variables $p_{b_i}$ and $q_{c_i}$ correspond to matching Reeb orbits. So we obtain:

$$= \pm \sum_{\Gamma\subseteq B\cap C} \pm\kappa_\Gamma \cdot t^{I\cup J} q^{A\cup C'} p^{B'\cup D} \hbar^{g_1+g_2+\#\Gamma-2}$$

(from now on, we will omit the signs and assume that they will work out with the orientation on $\partial\overline{\mathcal{M}}_{g,r}$.) So the coefficients of the monomials

$$t^{I\cup J} q^{A\cup C'} p^{B'\cup D} \hbar^{g_1+g_2+\#\Gamma-2}$$

are given by

$$\frac{\kappa_\Gamma}{\kappa_A\kappa_B\kappa_C\kappa_D} \frac{1}{r_1!\#A!\#B!} \frac{1}{r_2!\#C!\#D!} \int_{\overline{\mathcal{M}}_{g_1,r_1}(A,B)/\mathbf{R}} \mathrm{ev}^*\Theta_I \int_{\overline{\mathcal{M}}_{g_2,r_2}(C,D)/\mathbf{R}} \mathrm{ev}^*\Theta_J$$

which is equal to

$$\frac{\kappa_\Gamma}{\kappa_A\kappa_B\kappa_C\kappa_D} \frac{1}{r_1!\#A!\#B!} \frac{1}{r_2!\#C!\#D!} \int_{\overline{\mathcal{M}}_{g_1,r_1}(A,B)/\mathbf{R}\times\overline{\mathcal{M}}_{g_2,r_2}(C,D)/\mathbf{R}} \mathrm{ev}^*\Theta_{I\cup J}$$

But we know that

$$\overline{\mathcal{M}}_{g_1,r_1}(A,B)/\mathbf{R} \times \overline{\mathcal{M}}_{g_2,r_2}(C,D)/\mathbf{R}$$

is a $\kappa_\Gamma$-fold cover of a co-dimension one boundary stratum of

$$\overline{\mathcal{M}}_{g,r}(A\cup C', B'\cup D)/\mathbf{R}$$

(where $g = g_1 + g_2 + \#\Gamma - 1$, $r = r_1 + r_2$). There are other boundary strata consisting of disconnected curves (see the footnote on page 34). Over these, however, the integral of $\mathrm{ev}^*\Theta_I$ vanishes because of the extra $\mathbf{R}$-symmetry. So all *contributing* boundary strata will occur in $\mathbf{H}\cdot\mathbf{H}$, giving the following formula:

$$\mathbf{H}\cdot\mathbf{H} = \frac{1}{\hbar}\sum_{g}\sum_{A,r,s^\pm}\sum_{I,\Gamma^\pm}\frac{1}{\kappa_{\Gamma^-}\kappa_{\Gamma^+}}\frac{1}{r!s^-!s^+!}\int_{\partial\overline{\mathcal{M}}^A_{g,r}(\Gamma^-,\Gamma^+)/\mathbf{R}} \mathrm{ev}^*\Theta_I\, t^I q^{\Gamma^-} p^{\Gamma^+} z^A \hbar^g$$

Applying Stokes' theorem, we obtain

$$\mathbf{H}\cdot\mathbf{H} = 0$$

which is what we set out to prove. □

We define an operator $d^{\mathbf{H}}\colon \mathfrak{W}\to\mathfrak{W}$ given by $d^{\mathbf{H}}(a) = [\mathbf{H}, a]$. It is of degree $-1$ and by

$$\begin{aligned} d^{\mathbf{H}}\circ d^{\mathbf{H}} &= [\mathbf{H},[\mathbf{H},\cdot]] \\ \text{(by the Jacobi identity)} &= 2[[\mathbf{H},\mathbf{H}],\cdot] = 0 \end{aligned}$$

it is a chain operator. We write $H_{\mathrm{SFT}}(V,\alpha)$ for the homology of the pair $(\mathfrak{W}(V,\alpha), d^{\mathbf{H}})$.

We will also be interested in the semi-classical approximation to $\frac{1}{\hbar}\mathfrak{W}$. We define $\mathfrak{P} := \hbar\left((\frac{1}{\hbar}\mathfrak{W})/\mathfrak{W}\right)$ with Poisson bracket given by $\{a,b\} = \frac{i}{\hbar}[a,b]|_{\hbar\mapsto 0}$. We write $\mathbf{h}$ for the image of $\mathbf{H}$ in $\mathfrak{P}$. Then $d^{\mathbf{h}} := \{\mathbf{h},\cdot\}$ is a chain operator and we write $H_{\mathrm{rat}}(V,\alpha)$ for the homology of $(\mathfrak{P}(V,\alpha), d^{\mathbf{h}})$. Here, "rat" refers to the fact that $\mathbf{h}$ corresponds to the *rat*ional curves ($g = 0$).

## 4.3 POTENTIAL ASSOCIATED TO A COBORDISM

We now repeat what was done in the last section to construct a similarly defined power series associated to any (not necessarily cylindrical) cobordism. As before, we will write $W = \overrightarrow{V^-V^+}$.

1. We fix a *basic system of forms*

   $$(\Theta_1,\cdots,\Theta_{m+k}),$$

   meaning that

   a) these forms are linearly independent;

   b) the forms have cylindrical ends. This means that in the ends $V^\pm\times(\pm K,\pm\infty)$ (for large $K$) they are pullbacks of forms on $V^\pm$;

   c) the cohomology classes of $\Theta_1,\cdots,\Theta_m$ generate $H^\bullet(W)$;

d) and the cohomology classes of $\Theta_{m+1}, \cdots, \Theta_{m+k}$ generate the kernel of the map $H^\bullet_{\text{comp}}(W) \to H^\bullet(W)$. Here, $H_{\text{comp}}$ means homology with compact support.

We introduce formal variables $t^1, \cdots, t^{m+k}$ associated to these forms.

2. We introduce formal variables $q_\gamma$ for every Reeb orbit $\gamma$ in $V^-$, and formal variables $p_\gamma$ for every Reeb orbit $\gamma$ in $V^+$.

3. We fix a basis $(A^1, \cdots, A^j)$ for $H_2(W, \mathbf{Z})$ and to $A = d_i A^i \in H_2(W, \mathbf{Z})$ we associate a monomial $z_1^{d_1} \cdots z_j^{d_j}$ (we will also write $z^A$ for short).

We give variables a grading by the same formulae as in (4.2). We also introduce maps $\mathrm{ev}_i \colon \mathcal{M}^A_{g,r}(\Gamma^-, \Gamma^+) \to W$, sending a stable curve $C$ to the image in $W$ of its $i$th marked point, and all these maps together give

$$\mathrm{ev} \colon \mathcal{M}^A_{g,r}(\Gamma^-, \Gamma^+) \to \overbrace{W \times \cdots \times W}^{r \text{ times}}.$$

Then the symplectic field theory potential $\mathbf{F}$ is given by essentially the same formula as the one for $\mathbf{H}$. Here it is:

$$\mathbf{F} = \frac{1}{\hbar} \sum_g \sum_{A,r} \sum_{I,\Gamma^\pm} \frac{1}{r!s^-!s^+!} \frac{1}{\kappa_{\Gamma^-}\kappa_{\Gamma^+}} \int_{\overline{\mathcal{M}}^A_{g,r}(\Gamma^-,\Gamma^+)} \mathrm{ev}^*(\Theta_I)\, t^I q^{\Gamma^-} p^{\Gamma^+} z^A \hbar^g$$

Note that the pull-back extends smoothly to the boundary of $\overline{\mathcal{M}}^A_{g,r}$ because we assumed the forms to have cylindrical ends. By the dimension formula, $\mathbf{F}$ is homogeneous of degree 0.

We mention the following analogue of the master equation, proved in a similar way:

$$\overrightarrow{\mathbf{H}^-} e^{\mathbf{F}} - e^{\mathbf{F}} \overleftarrow{\mathbf{H}^+} = 0$$

where $\mathbf{H}^\pm$ are the Hamiltonians of the contact manifolds $V^\pm$, and where $\overrightarrow{\cdot}$ means substituting $p^- \mapsto \overrightarrow{\frac{\partial}{\partial q}}$,and similarly $\overleftarrow{\cdot}$ means $q^+ \mapsto \overleftarrow{\frac{\partial}{\partial p}}$.

**Proposition 4.3.1.**

## 4.4 CALCULATIONS FOR TARGET CURVES

Let us try to calculate some Hamiltonians and potentials explicitly.

**Example 4.4.1.** We calculate the Hamiltonian $\mathbf{H}$ for the contact manifold $S^1$ of dimension $2n-1$ for $n=1$. Let us choose a local coordinate $\phi$ such that $\int_{S^1} \mathrm{d}\phi = 1$, and let $\mathrm{d}\phi$ be the contact form. Then the Reeb flow is just $\frac{\partial}{\partial \phi}$ and the periodic orbits $\mathcal{P}$ are just $k$-fold covers ($k \geq 1$) of the circle. All Conley-Zehnder indices are zero because the symplectic path induced by an orbit is in the trivial group $\mathrm{Sp}(0)$. The cohomology $H^\bullet(S^1)$ is equal to $\langle 1, \mathrm{d}\phi \rangle$. The symplectization $W = S^1 \times \mathbf{R}$ has Chern class 0, so we ignore the $z$ variable.

We introduce variables $p_k$ and $q_k$ associated to the orbit of multiplicity $k$, and variables $t$ and $\tau$ associated to 1 and $\mathrm{d}\phi$ – we choose this notation for compatibility with what we will do later. We have deg $p_k$ = deg $q_k = -2$, deg $t_0 = -2$ and deg $t_1 = -1$. We are trying to compute (omitting the combinatorial factors)

$$\int_{\mathcal{M}_{g,r}(\Gamma^-,\Gamma^+)/\mathbf{R}} \mathrm{ev}^* \left( 1^{\otimes r-\ell} \otimes \mathrm{d}\phi^{\otimes \ell} \right) t^{r-\ell} \tau^\ell q^{\Gamma^-} p^{\Gamma^+} \hbar^g$$

for different $g, r$ and moduli spaces $\mathcal{M}_{g,r}(\Gamma^-, \Gamma^+)/\mathbf{R}$. Such a moduli space has dimension

$$\dim \mathcal{M}_{g,r}(s^-, s^+) = -5 + 4g + 2s^- + 2s^+ + 2r$$

So all these moduli spaces have odd dimension. Also, $\tau$ is an odd variable so $\tau^2 = 0$, which gives that we are only interested in $\ell \leq 1$. So we need only calculate contributions for moduli spaces of dimension 1.

We list the possible values of $g, r, s^\pm$.

1. $g = 1, r = 1, s^\pm = 0$. These are constant curves of genus 1. The moduli space is just $S^1$, corresponding to all point images in $W/\mathbf{R} = V$. The monomial is $\tau\hbar$ and because of virtual cycle complications the integral over the moduli space is $-\frac{1}{24}$. We obtain a contribution $-\frac{1}{24}\tau\hbar$.

2. $g = 0, r = 1, s^\pm = 1$. These are $k$-fold covers of the cylinder, asymptotically cylindrical over $q_k$ and $p_k$. The moduli space $\mathcal{M}_{0,1}(\{\gamma_k\}, \{\gamma_k\})/\mathbf{R}$ is, for every $k$, equal to $\sqcup_{i=1}^k S^1$, where every point in a copy of $S^1$ corresponds to a position of the marked point relative to that of the asymptotic markers, and where the different copies of $S^1$ correspond to the different configurations of the asymptotic markers. Also, $\mathrm{ev}^* \mathrm{d}\phi = k \cdot \mathrm{d}\phi$. We obtain a contribution $\tau \sum_{k=1}^\infty q_k p_k$.

3. $g = 0, r = 3, s^\pm = 0$. These are constant rational curves. The moduli space is just $S^1$, corresponding to all point images. Because of the combinatorics we scale by $\binom{r}{\ell}\frac{1}{r!} = \frac{3}{6}$. We obtain a contribution $\frac{1}{2}t^2\tau$.

Putting it all together:

$$\mathbf{H}(t \cdot 1 + \tau \cdot \mathrm{d}\phi) = \frac{1}{\hbar}\left(\frac{1}{2}t^2\tau + \tau \sum_{k=0}^{\infty} q_k p_k - \frac{1}{24}\tau\hbar\right)$$

**Example 4.4.2.** We now calculate the potential $\mathbf{F}$ for the complex plane $W = \mathbf{C}$, which can be seen as a symplectic cobordism with symplectic form that is a smooth extension of $\mathrm{d}r \wedge \mathrm{d}\phi$ to the unit disk, positive cylindrical end $V^+ = S^1$ and no negative cylindrical end. As a system of basic forms, we take $\Delta = 1$ (restricting to $\delta = 1$ on $S^1$), and $\Theta = \theta \wedge \mathrm{d}\phi$ for some compactly supported, rotationally invariant 1-form $\theta$, such that $\int_W \Theta = 1$. We have $c_1([W]) = 1$.

We introduce variables $t$ corresponding to 1, $\tau$ corresponding to $\Theta$, and $p_k$ corresponding to the Reeb orbits.

A curve that is asymptotically cylindrical over orbits $\gamma_{k_1}, \cdots, \gamma_{k_{s^+}}$ will be a map of degree $k_1 + \cdots + k_{s^+}$. Then $c_1(A)$ will be equal to the degree $k := k_1 + \cdots + k_{s^+}$. So The moduli space $\overline{\mathcal{M}}_{g,r}(\emptyset, \{\gamma_{k_1}, \cdots, \gamma_{k_{s^+}}\})$ has dimension

$$\dim \overline{\mathcal{M}}_{g,r}(0, s^+) = -4 + 4g + 2s^+ + 2r + 2k$$

We integrate a form of degree $\leq 2r$, so we are interested in the cases that this dimension is $\leq 2r$. We list them:

1. $g = 1$, $s^+ = k = 0$. These are constant curves of genus 1, so we require $r \geq 1$. The moduli space is $\overline{\mathcal{M}}_{1,r}(\{\mathrm{pt}\}) \times W$, where $W$ corresponds to the constant image. In this case, the map ev is equal on all $r$ factors, so the pull-back $\mathrm{ev}^*\Theta^{\otimes r}$ integrates to zero whenever $r > 1$. So the only contribution is $\tau\hbar$, which must be scaled by $-\frac{1}{24}$ because of virtual fundamental cycle complications.

2. $g = 0$, $s^+ = k = 0$. These are constant curves of genus 0, so $r \geq 3$. The moduli space is $\overline{\mathcal{M}}_{0,r}(\{\mathrm{pt}\}) \times W$. Again, ev is equal on all $r$ factors, so the pull-back $\mathrm{ev}^*\Delta^{\otimes 2} \otimes \Theta^{\otimes r-2}$ integrates to zero whenever $r > 3$. So the only contribution is $\frac{1}{2}t^2\tau$.

3. $g = 0$, $s^+ = 1$, $k = 1$. These are curves $f\colon \mathbf{C}P^1 \backslash \{\text{puncture}\} \to W$, so we may as well write $f\colon \mathbf{C} \to \mathbf{C}$ with a simple pole at $\infty$. Then the uncompactified moduli space is $(\mathbf{C}^r \backslash \Delta)$. Integrating $\mathrm{ev}^*\Theta^{\otimes r}$ gives 1, so we obtain a contribution
$$\sum_{r \geq 0}^{\infty} \frac{1}{r!}\tau^r p_1 z = \mathrm{e}^{\tau} p_1 z\,.$$

Putting this all together gives

$$\mathbf{F}(t\Delta + \tau\Theta) = \frac{1}{\hbar}\left(\frac{1}{2}t^2\tau + \mathrm{e}^{\tau}p_1 z - \frac{1}{24}\tau\hbar\right).$$

## 4.5 NON-DISCRETE SPACE OF REEB ORBITS

We will need a slight extension of this algebraic formalism to the case where the stable Hamiltonian structure is a circle bundle $V \to W$ over a symplectic manifold $W$. In this case, every fibre corresponds to a Reeb orbit, so the set of Reeb orbits is not discrete. However, we can still define an SFT Hamiltonian as follows.

First of all, note that it is not possible to define the notion of an asymptotic marker on these Reeb orbits in a continuous way, because we do not know whether $V$ is trivial. We will therefore consider moduli spaces of stable curves without asymptotic markers. Also, writing $H^\bullet(W) = \langle e_1, \cdots, e_s\rangle$ for a basis of homogeneous elements of $W$'s cohomology, we replace the variables $p_\gamma, q_\gamma$ by variables $p_{1,\kappa}, \cdots, p_{s,\kappa}, q_{1,\kappa}, \cdots, q_{s,\kappa}$, with commutation relation

$$[p_{i,\kappa}, q_{j,\kappa}] = \kappa\eta^{ij} \tag{4.4}$$

We replace the sum over sets of orbits by a sum over a number of homology classes (similar to the sum over $i_1, \cdots, i_r$ for the marked points).

To be precise, we define a moduli space

$$\overline{\mathcal{M}}^A_{0,r}(s^-, s^+, V)/\mathbf{R}$$

containing stable curves with $s^+ + s^-$ punctures, asymptotically cylindrical over some Reeb orbit. We define the map ev with maps $\mathrm{ev}_{p_1}, \cdots, \mathrm{ev}_{q_s}$ to obtain

$$\mathrm{ev}\colon \overline{\mathcal{M}}^A_{g,r}(s^-, s^+; V)/\mathbf{R} \to \overbrace{V \times \cdots \times V}^{r \text{ times}} \times \overbrace{W \times \cdots \times W}^{s^- + s^+ \text{ times}}.$$

Now let us write $\Theta_I = \Theta_{i_1} \cdots \Theta_{i_r}$, a differential form on $V$, and $e_{I^\pm} = e_{i_1} \cdots e_{i_{s^\pm}}$, a differential form on $W$. Then we define

$$\mathbf{H} := \frac{1}{\hbar}\sum_g \sum_{A,r,s^\pm} \sum_{I,I^\pm} \frac{1}{r!s^-!s^+!}\frac{1}{\kappa_{\Gamma^-}\kappa_{\Gamma^+}} \int_{\overline{\mathcal{M}}^A_{g,r}(s^-,s^+)/\mathbf{R}} \mathrm{ev}^*(\Theta_I e_{I^-} e_{I^+})\, t^I q^{\Gamma^-} p^{\Gamma^+} z^A \hbar^g$$

# 5

# GRAVITATIONAL DESCENDANTS

## 5.1 GROMOV-WITTEN POTENTIAL WITH DESCENDANTS

Let $(W, \omega, J)$ be a closed symplectic manifold. We write $H^\bullet(W) = \langle e_1, \cdots, e_s \rangle$ for a basis of homogeneous elements of its cohomology, with $e_1 = 1$ its unit. Let $\overline{\mathcal{M}}^A_{0,r}(W)$ be the moduli space of stable rational curves, realising homology class $A$, with $r$ marked points.

We define $r$ line bundles $\mathcal{L}_1, \cdots, \mathcal{L}_r$ over the moduli space $\overline{\mathcal{M}}^A_{0,r}(W)$ as follows. For the bundle $\mathcal{L}_i$, we want the fibre at a point $(C, y_1, \cdots, y_r) \in \overline{\mathcal{M}}^A_{0,r}(W)$ to be equal to $T^\vee|_{y_i}C$, the cotangent space to $C$ at the $i$th marked point. To define how these fibres fit together as a bundle, we consider a section $\sigma_i \colon \overline{\mathcal{M}}^A_{0,r}(W) \to \overline{\mathcal{M}}^A_{0,r+1}(W)$ of the bundle $\pi \colon \overline{\mathcal{M}}^A_{0,r+1}(W) \to \overline{\mathcal{M}}^A_{0,r}(W)$, where $\sigma_i$ sends $(C, y_1, \cdots, y_r)$ to the point $y_i$ in the fibre $\pi^{-1}(C) \cong C$. We define $\mathcal{L}_i$ as the pullback by $\sigma_i$ of the vertical component (i.e. $(\ker d\pi)^\vee$) of $T^\vee \overline{\mathcal{M}}^A_{0,r+1}(W)$.

We write $\psi_i$ for their first Chern classes $c_1(\mathcal{L}_i) \in H^2(\overline{\mathcal{M}}^A_{0,r}(W))$. We refer to these cohomology classes as *psi-classes*, or as *gravitational descendants*. We write $[\psi_i]$ for its Poincaré dual.

**Definition 5.1.1.** We introduce formal graded variables $t^{d,i}$ of degree $\deg e_i - 2(1 - d)$ and define the *rational Gromov-Witten potential of $W$*, $\mathbf{f}_W$, as the expression

$$\mathbf{f}_W := \sum_A \sum_r \sum_{I, d_1, \cdots, d_r} \int_{\overline{\mathcal{M}}^A_{0,r}(W)} \mathrm{ev}^*(e_{i_1} \cdots e_{i_r}) \psi_1^{d_1} \cdots \psi_r^{d_r} \frac{t^{d_1 i_1} \cdots t^{d_r i_r}}{r!} z^A$$

The integrals involving the cohomology classes $\psi_i$ are well-defined, because $\overline{\mathcal{M}}^A_{0,r}(W)$ has empty boundary. By the dimension formula, $\mathbf{f}$ is homogeneous of degree 0. We refer to the variables $t^{d,i}$ with $d > 0$ as *descendant variables.*

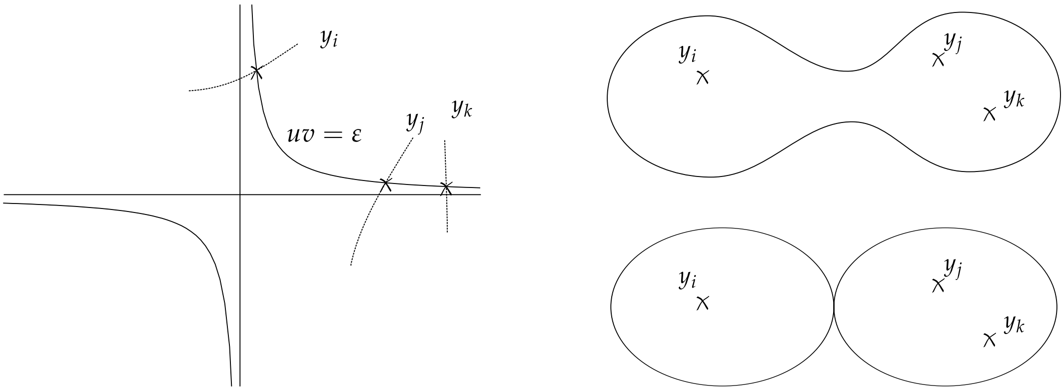



Figure 5.1: The degeneration of a smooth curve into a nodal curve. To the left is a half-dimensional 'cartoon'; to the right is a full-dimensional illustration. Note that the two irreducible components are not, in fact, tangent; on the right hand side, this illusion is created by the necessity of embedding in three dimensions.

DESCENDANTS WITH $d_i = 1$ It was noted by Witten in [14] that the cycles $[\psi_i]$ can be described as sets of curves having a certain configuration of nodes and marked points. Namely, if we fix distinct $1 \leq i, j, k \leq r$, then a representative of $[\psi_i]$ is given by the cycle that is the closure of the set of curves with two components, such that $y_i$ is on one component and $y_j$ together with $y_k$ on the other. The argument is as follows.

To calculate the divisor corresponding to $\psi_i = c_1(\mathcal{L}_i)$, we should choose a generic section of $\mathcal{L}_i$ and find its zero set. We first choose $j, k$ such that $1 \leq i, j, k \leq r$ are distinct. We then define the following section $s_{i;j,k}$. For $(C, y_1, \cdots, y_r)$ where $C$ has no nodes (so $C \cong \mathbf{C}P^1$), consider the differential form $\omega^C_{j,k}$ on $C$ that has poles at $y_j$ and $y_k$ with residues $+1$ and $-1$ respectively. Then $\omega^C_{j,k}$ has no zeroes, because the cotangent bundle of $\mathbf{C}P^1$ has degree 2. We let $s_{i;j,k}(C) := \omega^C_{j,k}|_{y_i}$.

So in other words, $s_{i;j,k}$ has no zeroes where $C$ has no nodes. Now let us see what happens to $\omega^C_{j,k}$ when $C$ degenerates into a nodal curve. We realise the degeneration as follows (and as illustrated in figure 5.1). Let $C_\varepsilon$ be the plane curve $uv = \varepsilon$ for $\varepsilon$ near $0 \in \mathbf{C}$. Then $C_\varepsilon$ is rational for every $\varepsilon \neq 0$ and it has two rational components for $\varepsilon = 0$, which are connected by a node at the origin $(u, v) = (0, 0)$.

If $\omega_\varepsilon$ is a family of meromorphic differential forms, say $\omega_\varepsilon = f_\varepsilon(u)\,\mathrm{d}u$, then it degenerates into $f_0(u)\,\mathrm{d}u$ on the component $v = 0$ and into $\lim_{\varepsilon \to 0} f_0(\varepsilon/v)\,{-\varepsilon}/v^2 \mathrm{d}v$ on the component $u = 0$. This limit only makes sense if $f_0$ has a pole of order no greater than 1 at the origin. We see that if $f_0$ has such a pole, then $\omega_0$ will have a pole of order 1 at the node on the first component, and also a pole of order 1 at the node on the second component. These poles will have opposite residue. Also, if $f_0$ has no pole at $u = 0$, then $\omega_0$ will be identically zero on the second component. We will consider such degenerate meromorphic differentiable forms to be the allowable differential forms on the degenerate curve. Note that it is a simple zero as a function of $\varepsilon$.

This means in particular that, for any nodal curve and two given non-nodal points on it, we can uniquely find an allowable meromorphic differential form having only poles at these two points with given (opposite) residues, and at the nodes. Indeed, the differential form must, on every component, either have two poles or be identically zero, which fixes it completely.

This describes in what way $\omega^C_{j,k}$ degenerates. Namely, when $y_i$ is on one component of $C$ and $y_j$ and $y_k$ on the other, then $\omega^C_{j,k}$ is identically zero on the first component and so $s_{i;j,k}(C) = 0$. When $y_i$ and $y_j$ are on one component and $y_k$ on the other, then $s_{i;j,k}(C) \neq 0$. So, as promised, we have that $[\psi_i]$ is the closure of the set of curves with two components, such that $y_i$ is on one component and $y_j$ together with $y_k$ on the other.

FORGETTING MARKED POINTS Considering this description of a psi class, we can easily see the following relation between $[\psi_i] \subseteq \overline{\mathcal{M}}^A_{0,r+1}(W)$ and $[\psi_i] \subseteq \overline{\mathcal{M}}^A_{0,r}(W)$:

$$[\psi_i] = \pi^* [\psi_i] + D_{i,r+1} \tag{5.1}$$

where $D_{i,r+1} \subseteq \overline{\mathcal{M}}^A_{0,r+1}(W)$ is the cycle of curves which have a constant component that contains only the marked points $y_i$ and $y_{r+1}$ together with a node. Indeed, these are precisely the curves in $[\psi_i] \subseteq \overline{\mathcal{M}}^A_{0,r+1}(W)$ whose topologies are modified when forgetting $y_{r+1}$ in such a way that they do not end up in $[\psi_i] \subseteq \overline{\mathcal{M}}^A_{0,r}(W)$. This is known as the *comparison formula.*

Let us write $\sigma_i \colon \overline{\mathcal{M}}_{0,r} \to \overline{\mathcal{M}}_{0,r+1}$ for the map sending a curve to the $i$th marked point in $\overline{\mathcal{M}}_{0,r+1}$ (interpreted as its universal curve). Then we see that

$$D_{i,r+1} = \operatorname{im} \sigma_i$$

More generally, we have

$$D^{d_i}_{i,r+1} = (-1)^{d_i - 1}\, \sigma_{i*}\, [\psi_i]^{d_i - 1} \tag{5.2}$$

which we will show now. An intersection $D_{i,r+1} \cdot D^{d_i-1}_{i,r+1}$ is defined by generically perturbing the cycle $D^{d_i-1}_{i,r+1}$ into a cycle $E$ and then taking $D \cdot E$. Note that $D^{d_i-1}_{i,r+1} \subseteq$

$D_{i,r+1}$. So infinitesimally, a generic perturbation $E$ is given by a generic section of the bundle

$$N := \left(T\overline{\mathcal{M}}^A_{0,r+1}(W)/TD\right)|_{D^{d_i-1}_{i,r+1}}$$

over $D^{d_i-1}_{i,r+1}$. Then the intersection $D \cdot E$ is given by the maximal Chern class $c_{\max}(N)$ in the cohomology of $D^{d_i-1}_{i,r+1}$.

So let us describe $N$. Locally near a nodal curve $C = C_1 \vee C_2 \in D$, a neighbourhood of $D$ is given by curves $uv = \varepsilon$ (see again fig. 5.1). Then an infinitesimal change in $\varepsilon$ can be regarded an element of $T_{u=0}C_1 \otimes T_{v=0}C_2$, the tensor product of the tangent spaces at the node. Now let us regard the two components $(C_1, C_2)$ as elements of moduli spaces $\overline{\mathcal{M}}^0_{0,3} \times \overline{\mathcal{M}}^A_{0,r}$, with the additional marked points corresponding to the node. Then as a bundle on $\overline{\mathcal{M}}_{0,r}$, the tangent space $T_{v=0}C_2$ is dual to $\mathcal{L}_i$, so its maximal Chern class is just $-\psi_i$. Similarly, $T_{u=0}C_1$ is dual to $\mathcal{L}_3 \to \overline{\mathcal{M}}^0_{0,3}$, which is trivial. Then

$$c(N) = -\sigma_{i*}\psi_i$$

Intersecting with $D^{d_i-1}_{i,r+1}$, we obtain (5.2) inductively.

## 5.2 STRING EQUATION AND TOPOLOGICAL RECURSION RELATIONS

Our description of the descendant moduli spaces can be seen as a description of these divisors in terms of lower-dimensional moduli spaces. The exact relation is best expressed as a set of partial differential equations for the rational Gromov-Witten potential. They involve the Poincaré pairing $\eta_{ij} := \int_W e_i e_j$; we also write $\eta^{ij}$ for the coefficients of the inverse matrix. The equations are the following:

**Proposition 5.2.1** (The string equation (SE)). *We have:*

$$\frac{\partial \mathbf{f}}{\partial t^{0,1}} = \sum_{d,i} t^{d+1,i} \frac{\partial \mathbf{f}}{\partial t^{d,i}} + \frac{1}{2} \sum_{i,j} t^{0,i} \eta_{ij} t^{0,j}$$

*Proof.* The last term on the right hand side accounts for constant curves with three marked points, which have no image when trying to define a map $\pi \colon \overline{\mathcal{M}}^A_{0,3}(W) \to \overline{\mathcal{M}}^A_{0,2}(W)$: the coefficient of the corresponding monomial $t^{0,1}t^{0,i}t^{0,j}$ is given by

$$\int_{\{\text{constant curves}\}} \mathrm{ev}^* e_1 e_i e_j = \int_W e_1 e_i e_j = \eta_{ij}$$

The rest comes down to proving the following formula

$$\int_{\overline{\mathcal{M}}^A_{0,r+1}(W)} \mathrm{ev}^*(\alpha \cdot e_1)\psi_1^{d_1}\cdots\psi_r^{d_r}\psi_{r+1}^0 = \sum_{i=1}^r \int_{\overline{\mathcal{M}}^A_{0,r}(W)} \mathrm{ev}^*(\alpha)\psi_1^{d_1}\cdots\psi_i^{d_i-1}\cdots\psi_r^{d_r}$$

(where, on the RHS, terms with negative exponents should be ignored). Note that $e_1$ is just 1 so we can ignore it. By the comparison formula (5.1) we have

$$\pi^*\psi_i^{d_i} \quad = \quad \left([\psi_i]^{\cdot d_i} - D_{i,r+1}\right)^{d_i}$$

Now if we represent $[\psi_i]$ by the zero set of $s_{i;j,r+1}$ for some $j \notin \{i, r+1\}$, then it is clear that the cross-terms give empty intersection. So we obtain

$$\begin{aligned} &= \quad [\psi_i]^{\cdot d_i} + (-1)^{d_i} D_{i,r+1}^{\cdot d_i} \\ \text{(by (5.2))} \quad &= \quad [\psi_i]^{\cdot d_i} + (-1)^{d_i}(-1)^{d_i-1}\sigma_{i*}\left[\psi_i\right]^{(d_i-1)} \\ &= \quad [\psi_i]^{\cdot d_i} - \sigma_{i*}\left[\psi_i\right]^{(d_i-1)} \end{aligned}$$

We then see

$$\begin{aligned} \int_{\overline{\mathcal{M}}_{0,r+1}} \mathrm{ev}^*(\alpha)\psi_1^{d_1}\cdots\psi_r^{d_r} \quad &= \quad \int_{\overline{\mathcal{M}}_{0,r+1}} \mathrm{ev}^*(\alpha)\left(\pi^*\psi_1^{d_1} + \sigma_{1*}\psi_1^{d_1-1}\right) \\ &\qquad \cdots\left(\pi^*\psi_r^{d_r} + \sigma_{r*}\psi_r^{d_r-1}\right) \end{aligned}$$

The term $\pi^*(\psi_1^{d_1}\cdots\psi_r^{d_r})$ does not contribute for dimensional reasons, and neither do terms with more than one factor $\sigma_{i*}\psi_i^{d_i-1}$ because the cycles $D_{i,r+1}$ do not intersect each other for different $i$. So we obtain

$$\begin{aligned} &= \quad \sum_{i=1}^r \int_{\overline{\mathcal{M}}_{0,r+1}} \mathrm{ev}^*(\alpha)\pi^*\psi_1^{d_1}\cdots\sigma_{i*}\psi_i^{d_i-1}\cdots\pi^*\psi_r^{d_r} \\ &= \quad \sum_{i=1}^r \int_{\overline{\mathcal{M}}_{0,r}} \mathrm{ev}^*(\alpha)\psi_1^{d_1}\cdots\psi_i^{d_i-1}\cdots\psi_r^{d_r} \end{aligned}$$

(here we also use that $\mathrm{ev}_i \circ \pi = \mathrm{ev}_i$), which is what we wanted to prove. □

**Proposition 5.2.2** (The topological recursion relations (TRR)). *For any indices $\alpha, \beta, \gamma$ and descendant levels $d, b, c$, we have:*

$$\frac{\partial^3 \mathbf{f}}{\partial t^{d+1,\alpha}\partial t^{b,\beta}\partial t^{c,\gamma}} = \sum_{i,j} \frac{\partial^2 \mathbf{f}}{\partial t^{d,\alpha}\partial t^{0,k}}\eta^{k\ell}\frac{\partial^3 \mathbf{f}}{\partial t^{0,\ell}\partial t^{b,\beta}\partial t^{c,\gamma}}$$

*Proof.* Suppose that $t^{d_1,i_i} \cdots t^{d_r i_r}$ is a term of the LHS, so its coefficient is an integral over the moduli space $\overline{\mathcal{M}}^A_{0,r+3}(W)$. More precisely, it is a sum over all ways of assigning indices $\alpha, \beta, \gamma$ (and their corresponding descendant levels $d+1, b, c$) to the $r+3$ marked points. The form to be integrated contains a certain $\psi_i$ with positive exponent $d+1$, so we may instead integrate over $[\psi_i]$ and replace the exponent by $d$.

Now, components of $[\psi_i]$ can be identified with partitions $X \sqcup Y = \{1, \cdots, r\}$ A curve $C_1 \vee C_2$ in such a component, for which we will write $[X \mid Y]$, can be regarded as an element of $\overline{\mathcal{M}}^{A_1}_{0,\#X+1}(W) \times \overline{\mathcal{M}}^{A_2}_{0,\#Y+1}(W)$. Let us write $\tau\colon [X \mid Y] \subseteq [\psi_i] \to \overline{\mathcal{M}}^{A_1}_{0,\#X+1}(W) \times \overline{\mathcal{M}}^{A_2}_{0,\#Y+1}(W)$. Then the image of $\tau$ is exactly the pairs of curves for which the extra marked points map to the same point in $W$.

With this notation, the topological recursion relations come down to proving

$$\int_{[\psi_i] \subseteq \overline{\mathcal{M}}^A_{0,r}} \mathrm{ev}^*(\alpha)\psi_1^{d_1} \cdots \psi_r^{d_r} =$$

$$\sum_{\substack{A_1+A_2=A \\ r_1+r_2=r}} \int_{\overline{\mathcal{M}}^{A_1}_{0,r_1+1}(W)} \mathrm{ev}^*(\alpha e_i)\psi_1^{d'_1} \cdots \psi_{r_1}^{d'_{r_1}} \eta^{ij} \int_{\overline{\mathcal{M}}^{A_2}_{0,r_2+1}(W)} \mathrm{ev}^*(\alpha e_j)\psi_1^{d''_1} \cdots \psi_{r_2}^{d''_{r_2}}$$

Here, the exponents of the psi classes are just given by distributing $d_1, \cdots, d_r$ over $d'_1, \cdots, d'_{r_1}, d''_1 \cdots, d''_{r_2}$ depending on the partition $X \cup Y$. The sum over homology classes $A_1 + A_2 = A$ has only finitely many nonzero terms, because the moduli spaces are empty when $\omega(A) < 0$.

When passing from the psi-class cycle to to product of moduli spaces in this way, the only difficulty is in imposing the constraint that the extra marked points map to the same point in $W$. For this, we consider the map

$$\mathrm{ev}_{r_1+1} \times \mathrm{ev}_{r_2+1}\colon \overline{\mathcal{M}}_{0,r_1+1} \times \overline{\mathcal{M}}_{0,r_2+1} \to W \times W$$

We write $\Delta \subseteq W \times W$ for the diagonal. Then $\left(\mathrm{ev}_{r_1+1} \times \mathrm{ev}_{r_2+1}\right)^{-1}(\Delta)$ is precisely the image of $\tau$. In cohomology, the Poincaré dual to $\Delta$ is given by the Kunneth formula by

$$\eta^{ij} e_i \wedge e_j$$

(where we should regard the first/second wedge factor as cocycles in the first/second direct product factor). Then an integral over $\left(\mathrm{ev}_{r_1+1} \times \mathrm{ev}_{r_2+1}\right)^{-1}(\Delta)$ is just an integral of the form given above. This concludes the proof. □

## 5.3 GRAVITATIONAL DESCENDANTS IN SFT

In the context of symplectic field theory, the definition of gravitational descendants, and establishing similar PDE's for the SFT potential, is a work in progress. A definition of gravitational descendants is given in [3], and some work on establishing (generalisations of) the string equation and of the topological recursion relations has been done in [4, 5].

The main difficulty is that the SFT-moduli spaces have a topological boundary while Gromov-Witten moduli space have not. This means in particular that an integral over a Chern class, which is only defined up to addition of a closed form, depends on the chosen representative. Furthermore, the convenient line bundle sections that we chose to give a description of the divisors $[\psi_i]$ fail to be generic, since they will have zero sets of co-dimension 1. In the case of the string equation, one can work around these problems by considering it to be a relation in SFT-homology in stead of in the Weyl algebra itself, see [4]. (One must also ensure that the chosen representatives in different moduli spaces are in some way 'coherent'.) The second problem seems to be what is currently blocking generalising the topological recursion relations to SFT, although some work (for a very specific case and with a modification of the potential and moduli spaces) has been done in [5]. The authors suggest to consider, a modified symplectic field theory, which they call *non-equivariant*, which should correspond to how one can remove the additional $S^1$-symmetry in moduli spaces such at $\overline{\mathcal{M}}_{0,r}(\{\gamma\},\{\gamma^+\})/\mathbf{R}$. They give details in the case of cylindrical contact homology, which is basically SFT retricted to these moduli spaces.

For what follows, however, we will need no more than the definition of gravitational descendants from [3], and only in the cylindrical case $W = V \times \mathbf{R}$. The trick is to define *descendant moduli spaces* $\overline{\mathcal{M}}_{0,r}^{d_1,\cdots,d_r}(\Gamma^-,\Gamma^+;W)/\mathbf{R}$, which will play the role of the cycle $\left[\psi_1^{d_1}\cdots\psi_r^{d_r}\right]$. For this, we again introduce $r$ line-bundles $\mathcal{L}_i$ ($1 \le i \le r$) on $\overline{\mathcal{M}}_{0,r}(\Gamma^-,\Gamma^+;W)/\mathbf{R}$, given fibre-wise by the cotangent space at the $i$th marked point. If $\overline{\mathcal{M}}_{r_1}(\Gamma^-,\Delta^+;W)/\mathbf{R} \times \overline{\mathcal{M}}_{r_2}(\Delta^-,\Gamma^+;W)/\mathbf{R}$ describes a boundary stratum of $\overline{\mathcal{M}}_{0,r}(\Gamma^-,\Gamma^+;W)/\mathbf{R}$, then $\mathcal{L}_i$ restricted to this stratum is just the pull-back of the corresponding line bundle on either of the factors. We define a *coherent collection of sections* as a collection of sections for every line bundle $\mathcal{L}_i$ of every moduli space (for a given target manifold) such that these sections agree under this pull-back on the boundary stratum.

Given a coherent collection of sections, we define the descendant moduli spaces $\overline{\mathcal{M}}^{\cdots 0,1,0\cdots} = \overline{\mathcal{M}}_{0,r}^{A;\cdots 0,1,0\cdots}(\Gamma^-,\Gamma^+;W)/\mathbf{R}$ as the zero set of the section of $\mathcal{L}_i$. We next take the line bundles $\mathcal{L}_i^{\otimes 2}$ on these descendant moduli spaces and take a coherent collection of sections; we define the moduli spaces $\overline{\mathcal{M}}^{\cdots 0,2,0\cdots}$ as their zero set. Inductively, we define $\overline{\mathcal{M}}^{\cdots 0,d_i,0\cdots}$. We define

$$\overline{\mathcal{M}}^{d_1\cdots,d_r} := \overline{\mathcal{M}}^{d_1,0,\cdots} \cap \cdots \cap \overline{\mathcal{M}}^{\cdots 0,d_r}$$

which is generically a transversal intersection. Lastly, we define the rational descendant Hamiltonian, for which we also write $\mathbf{h}$ when no confusion is to be feared, as

$$\mathbf{h} := \sum_{A,r,s^\pm} \sum_{I,D,\Gamma^\pm} \frac{1}{r!s^-!s^+!} \frac{1}{\kappa_{\Gamma^-}\kappa_{\Gamma^+}} \frac{1}{d_1!\cdots d_r!} \int_{\overline{\mathcal{M}}_{0,r}^{A;d_1,\cdots,d_r}(\Gamma^-,\Gamma^+)/\mathbf{R}} \mathrm{ev}^*(\Theta_I)\, t^{I,D} q^{\Gamma^-} p^{\Gamma^+} z^A$$

Because we chose the sections to be coherent, we can easily prove this

**Proposition 5.3.1.** *The descendant Hamiltonian satisfies the master equation*

$$\{\mathbf{h}, \mathbf{h}\} = 0$$

*Proof.* This is proved in [3]. The proof is similar to, and no more difficult than, our proof of proposition 4.2.1. □

# 6

# INTEGRABLE SYSTEMS

## 6.1 INTRODUCTION

In physics, one tries to describe the evolution of a physical system as a path in an appropriate phase space. Let us call this phase space $\mathcal{L}$; it may have coordinates such as position, momentum, wave amplitude, et cetera. We are interested in the dynamics of function(al)s on $\mathcal{L}$ as time evolves. These dynamics are commonly expressed by endowing the space of functionals $\mathfrak{P}$ with a *Poisson bracket* $\{\cdot,\cdot\}_{\mathfrak{P}}$, by fixing a distinguished functional $H$, called the *Hamiltonian*[1], and by the equation

$$\frac{\partial F}{\partial t} = \{F, H\}_{\mathfrak{P}}$$

One approach to solving such a system is by finding *constants of motion*, or *symmetries*, which are functionals $F$ such that $F[u]$ is independent of $t$. It is clear that $H$ itself is such a constant of motion, since $\{H, H\}_{\mathfrak{P}} = 0$. For time-independence, we want these constants of motion to Poisson-commute with the Hamiltonian $H$, and for technical reasons, also with each other. In fact, it is common to start with trying to find a collection of Poisson-commuting functionals $F_p$, and then selecting a Hamiltonian among them, hopefully corresponding to some physical system. Systems for which a sufficient number (in a well-defined sense) of explicit independent commuting symmetries can be found are called *integrable systems*[2].

In the following, we will not distinguish between Hamiltonians (which dictate certain dynamics) and symmetries (which do not). Instead, we refer to all of them as Hamiltonians and introduce a 'time' variable for every one of them. The result is called an *integrable hierarchy*.

1 To avoid confusion later on, we should point out now that it will not be the SFT Hamiltonian $\mathbf{H}$, but rather its partial derivatives that will play this role.

2 Note that this is not a mathematically meaningful definition, because it is unclear what 'explicit' means.

## 6.2 INFINITE DIMENSIONAL HAMILTONIAN SYSTEMS

We will, following [11], define a Poisson algebra $\mathfrak{P}$ of functionals on a formal loop space $\mathcal{L}$. We will use Einstein's summation convention.

FUNCTIONALS Let $\mathcal{L} = \mathcal{L}(\mathbf{R}^n)$ be the space of smooth functions $u = (u^1, \cdots, u^n)\colon S^1 \to \mathbf{R}^n$. Also, let $\mathrm{e}^{\mathrm{i}x} \mapsto x$ be a coordinate on $S^1$. We write $u_x, u_{xx}, \cdots$ or $u^{(s)}$ for the derivatives with respect to $x$, and $u^{\alpha,s}$ for the $s$th derivative of the $\alpha$th component of $u$. Then a *local functional* $F$ on $\mathcal{L}$ is a map $\mathcal{L} \to \mathbf{C}$ given by an expression

$$F[u] = \frac{1}{2\pi} \int_0^{2\pi} f(x, u, u_x, u_{xx}, \cdots)\, \mathrm{d}x$$

where $f$ is a smooth function of $x$ and $u$ and polynomial in the derivatives $u_x, u_{xx}, \cdots$. We extend the set of these to a set $\mathfrak{P}$ of *formal* local functionals by regarding the variables $x, u, u_x, \cdots$ as formal independent variables. In other words, $\mathfrak{P}$ is the set of formal power series in $x, u, u_x, \cdots$ that are analytic in $x$ and $u$ and polynomial in the other variables, modulo total $x$-derivatives.

POISSON BRACKET We now introduce a notion of derivative for a local functional. If $F$ is a local functional, we can, for a perturbation $v \in T|_u\mathcal{L}$ represented by vectors $v(x) \in \mathbf{R}^n \cong T|_{u(x)}\mathbf{R}^n$, consider the form $\delta_v F[u]$ given by

$$\delta_v F[u] := \frac{\partial}{\partial \varepsilon}|_{\varepsilon=0} F[u + \varepsilon v]$$

A short calculation gives

$$\begin{aligned} \delta_v F[u] &= \frac{1}{2\pi} \int_0^{2\pi} \sum_{s=0}^{\infty} \frac{\partial f}{\partial u^{j,s}} v^{j,s}\, \mathrm{d}x \\ \text{(by integration-by-parts)} &= \frac{1}{2\pi} \int_0^{2\pi} \sum_{s=0}^{\infty} v^{j,0} (-\partial_x)^s \frac{\partial f}{\partial u^{j,s}}\, \mathrm{d}x \\ &=: \frac{1}{2\pi} \int^{2\pi} v^{j,0} \frac{\delta f}{\delta u^j}(u)\, \mathrm{d}x \end{aligned}$$

Given a flat metric $\eta_{ij}$ on $\mathbf{R}^n$, we will now introduce a Poisson bracket $\{\cdot,\cdot\}_{\mathfrak{P}}$. For this, it will be convenient to write $\{f,G\} = \{F,g\}$ for the density corresponding to the functional $\{F,G\}$ (where $f,g$ are the densities for $F,G$). We define:

$$\{f,G\}_{\mathfrak{P}} = \frac{1}{2\pi}\int_0^{2\pi} \frac{\delta f}{\delta u^i}\eta^{ij}\partial_x\frac{\delta g}{\delta u^j}\,\mathrm{d}x$$

FOURIER EXPANSION We can express the loop $(u, u_x, u_{xx}, \cdots)$ in Fourier coefficients $(u_0, q_k, p_k)$, defined by the equation

$$u^j = u_0^j + \sum_{k=1}^{\infty} q_k^j \mathrm{e}^{-\mathrm{i}kx} + p_k^j \mathrm{e}^{\mathrm{i}kx}$$

and its $x$-derivatives. We can regard $u_0^j, q_k^j, p_k^j$ as elements of $\mathfrak{P}$ with densities $u^j, u^j\mathrm{e}^{\mathrm{i}kx}, u^j\mathrm{e}^{-\mathrm{i}kx}$ respectively. So it makes sense to calculate their Poisson brackets. We have

$$\begin{aligned} q_k^j[u] \text{ gives } \frac{\delta\left(u^j\mathrm{e}^{\mathrm{i}kx}\right)}{\delta u^\ell} &= \delta_\ell^j \mathrm{e}^{\mathrm{i}kx} \\ p_k^j[u] \text{ gives } \frac{\delta\left(u^j\mathrm{e}^{-\mathrm{i}kx}\right)}{\delta u^\ell} &= \delta_\ell^j \mathrm{e}^{-\mathrm{i}kx} \end{aligned}$$

so

$$\begin{aligned} \{p_k^i, q_\ell^j\}_{\mathfrak{P}}[u] &= \int_{S^1} \mathrm{e}^{-\mathrm{i}kx}\eta^{ij}\partial_x \mathrm{e}^{\mathrm{i}\ell x}\,\mathrm{d}x \\ &= \mathrm{i}\ell\eta^{ij}\int_{S^1} \mathrm{e}^{\mathrm{i}(\ell-k)x}\mathrm{d}x \\ &= \mathrm{i}\ell\delta_{k\ell}\eta^{ij} \end{aligned}$$

For later use, we note now that this is the same formula as the semi-classical approximation to (4.4).

POISSON-COMMUTING Let us now see what it means for two functionals $F$ and $G$ to Poisson-commute in terms of their density. Remember that two functionals $F$ and $G$ *Poisson-commute* if and only if $\{F,G\} = 0$. That is,

$$\int_0^{2\pi} \{f,G\}(u)\,\mathrm{d}x = 0$$

for every $u \in \mathcal{L}$. This happens if and only if $\{f, G\}$ is a total $x$-derivative. Let us write $\{f, G\} = \partial_x \Omega$. Then explicitly

$$\begin{aligned} \partial_x \Omega &= \{f, G\} \\ &= \sum_{s,t=0}^{\infty} \frac{\partial f}{\partial u^{i,s}} \eta^{ij} \partial_x \frac{\partial g}{\partial u^{j,t}} \end{aligned} \tag{6.1}$$

Later on, we will be interested in the special case that $f$ and $g$ only depend on $u$ and not on its $x$-derivatives. Then we obtain

$$\begin{aligned} \partial_x \Omega &= \frac{\partial f}{\partial u^i} \eta^{ij} \partial_x \frac{\partial g}{\partial u^j} \\ \sum_m u_x^m \frac{\partial \Omega}{\partial u^m} &= \sum_m u_x^m \frac{\partial f}{\partial u^i} \eta^{ij} \frac{\partial^2 g}{\partial u^m u^j} \\ \frac{\partial \Omega}{\partial u^m} &= \frac{\partial f}{\partial u^i} \eta^{ij} \frac{\partial^2 g}{\partial u^m u^j} \qquad \text{for all } m \\ \mathrm{d}\Omega = \frac{\partial \Omega}{\partial u^m} \mathrm{d}u^m &= \frac{\partial f}{\partial u^i} \eta^{ij} \frac{\partial^2 g}{\partial u^m u^j} \mathrm{d}u^m \end{aligned}$$

So in particular, the form $\frac{\partial f}{\partial u^i} \eta^{ij} \frac{\partial^2 g}{\partial u^m u^j} \mathrm{d}u^m$ is closed. This means

$$\begin{aligned} \frac{\partial}{\partial u^n} \left( \frac{\partial f}{\partial u^i} \eta^{ij} \frac{\partial^2 g}{\partial u^m u^j} \right) &= \frac{\partial}{\partial u^m} \left( \frac{\partial f}{\partial u^i} \eta^{ij} \frac{\partial^2 g}{\partial u^n u^j} \right) \\ \frac{\partial^2 f}{\partial u^n \partial u^i} \eta^{ij} \frac{\partial^2 g}{\partial u^m u^j} &= \frac{\partial^2 f}{\partial^2 u^m \partial u^i} \eta^{ij} \frac{\partial^2 g}{\partial u^n u^j} \end{aligned} \tag{6.2}$$

We will refer back to here when we obtain commuting Hamiltonians from symplectic field theory.

## 6.3 THE DUBROVIN-ZHANG PRINCIPAL HIERARCHY – TAKE I

We have a particularly interesting way of finding a collection of symmetries in the case where we can describe the phase space as the loop space of the cohomology of a symplectic manifold, with Poisson bracket induced by the Poincaré pairing (considered as a flat metric). We will describe this construction in this section.

Consider the rational Gromov-Witten potential $\mathbf{f}(t^{0,1}, t^{0,2}, \cdots)$ for a symplectic manifold $M$, with cohomology generated by $\Delta_1, \cdots, \Delta_s$. Here, we will choose $\Delta_1$ to be the unit in the cohomology ring, and it will play a distinguished role in what follows. We define the pairing $\eta_{ij} = \int_M \Delta_i \Delta_j$. We consider $t^{0,1}, \cdots, t^{0,s}$ to be coordinates for $H^\bullet(M)$, and the $t^{d,j}$ are considered to be time variables for $j > 0$. We will often write $t$ for the set of $t^{d,i}$-variables. We label some of $\mathbf{f}$'s derivatives:

$$\Omega_{a\alpha b\beta}(t) := \frac{\partial^2 \mathbf{f}}{\partial t^{b\beta} \partial t^{a\alpha}}$$

and we define the function

$$\mathbf{u}^i(x,t) := \eta^{ij}\frac{\partial^2 \mathbf{f}}{\partial t^{0,j}\partial t^{0,1}}(x+t^{0,1},t^{0,2},\cdots)$$

which we regard as a $t$-dependent loop in $\mathcal{L}$. Note the peculiarity of what this means: for a given $x$, $\mathbf{u}(x)$ is both parametrised by $H^\bullet(M)$ (and also by descendant variables), and it takes values in it. This is motivated by the following

**Proposition 6.3.1.** *We have (substituting $x \mapsto 0$)*

$$\Omega_{a\alpha b\beta}(t) = \Omega_{a\alpha b\beta}(\mathbf{u}^1(t),\cdots,\mathbf{u}^s(t),0,0,\cdots)$$

*i.e. $\Omega_{a\alpha b\beta}$ is invariant under the substitution $t^{0,i} \mapsto \mathbf{u}^i(t)$ and $t^{d,i} \mapsto 0$ for $d > 0$.*

*Proof.* We show that $\Omega(t)$ and $\Omega(\mathbf{u}^1(t),\cdots,\mathbf{u}^s(t),0,0,\cdots)$ both solve the same system of ordinary differential equations with the same starting value. First of all, we have

$$\begin{aligned}
\mathbf{u}^i(t^{0,1},\cdots,t^{0,s},0,0,\cdots) &= \eta^{ij}\frac{\partial^2\mathbf{f}}{\partial t^{0,j}\partial t^{0,1}}|_{(t^{0,1},\cdots,t^{0,s},0,0,\cdots)} \\
\text{(by the string equation)} &= \eta^{ij}\frac{\partial}{\partial t^{0,j}}\left[\sum_{d,i} t^{d+1,i}\frac{\partial \mathbf{f}}{\partial t^{d,i}} + \frac{1}{2}t^{0,k}\eta_{k\ell}t^{0,\ell}\right]|_{(t^{0,1},\cdots,t^{0,s},0,0,\cdots)} \\
&= \eta^{ij}\left[\sum_{d,i} t^{d+1,i}\frac{\partial^2\mathbf{f}}{\partial t^{0,j}\partial t^{d,i}} + \eta_{j\ell}t^{0,\ell}\right]|_{(t^{0,1},\cdots,t^{0,s},0,0,\cdots)} \\
\text{(because all } t^{d+1,i}\text{are 0)} &= t^{0,i}
\end{aligned}$$

so we have

$$\Omega(t^{0,1},\cdots,t^{0,s},0,0,\cdots) = \Omega(\mathbf{u}^1(t^{0,1},\cdots,t^{0,s},0,0,\cdots),\cdots,\mathbf{u}^s(t^{0,1},\cdots,t^{0,s},0,0,\cdots),0,0,\cdots)$$

Next, consider the vector fields $X_{d,k}$ on the space parametrised by $t$, given by

$$X_{d,k} := \frac{\partial}{\partial t^{d+1,k}} - \sum_{i,j}\frac{\partial^2\mathbf{f}}{\partial t^{d,k}\partial t^{0,i}}\eta^{ij}\frac{\partial}{\partial t^{0,j}}\,.$$

The topological recursion relations yield directly

$$X_{d,k}\mathbf{u}^i = 0$$

and

$$X_{d,k}\Omega_{a\alpha b\beta} = 0$$

and so

$$X_{d,k}\Omega(\mathbf{u}^1(t),\cdots,\mathbf{u}^s(t),0,0,\cdots) = \sum_i \frac{\partial\Omega}{\partial t^{0,i}}|_{(\mathbf{u}^1(t),\cdots,\mathbf{u}^s(t),0,0,\cdots)} X_{d,k}\mathbf{u}^i = 0$$

So we have $X_{d,k}\Omega(t) = X_{d,k}\Omega(\mathbf{u}(t)) = 0$ for a system $\{X_{d,k}\}$ of vector fields. Then $\Omega(t) = \Omega(\mathbf{u}(t))$. □

We will define the *Hamiltonians*

$$\begin{aligned} h_{di}(u) &= \Omega_{d+1,i,0,1}(u^1,\cdots,u^s,0,0,\cdots) \\ \mathbf{h}_{di}[u] &= \frac{1}{2\pi}\int_0^{2\pi} h_{di}(u)\,\mathrm{d}x \end{aligned}$$

Then:

**Lemma 6.3.2.** *We have*

$$h_{di} = \frac{\partial\mathbf{f}}{\partial t^{d,i}}|_{(u^1,\cdots,u^s,0,0,\cdots)}$$

*Proof.* Apply the string equation. □

**Proposition 6.3.3.** *The function* $\mathbf{u}$ *solves the Hamiltonian system*

$$\frac{\partial\mathbf{u}^j}{\partial t^{di}} = \{u^j,\mathbf{h}_{di}\}(\mathbf{u})$$

*(here* $u^j$ *is considered a density and therefore so is* $\{u^j,\mathbf{h}_{di}\}$*, into which we substitute* $u^k\mapsto\mathbf{u}^k$*) and the Hamiltonians Poisson-commute:*

$$\{\mathbf{h}_{d,i},\mathbf{h}_{e,j}\} = 0$$

*Proof.* We calculate (writing $\delta_i = \frac{\delta}{\delta u^i}$ for short):

$$\begin{aligned} \{u^j,\mathbf{h}_{di}\}(\mathbf{u}^j) &= \delta_k u^j \eta^{k\ell}\partial_x\delta_\ell h_{di}(\mathbf{u}^1,\cdots,\mathbf{u}^s,0,0,\cdots) \\ &= \delta_k^j\eta^{k\ell}\partial_x\frac{\partial h_{d,i}}{\partial u^\ell}(\mathbf{u}^1,\cdots,\mathbf{u}^s,0,0,\cdots) \\ &= \eta^{j\ell}\partial_x\frac{\partial^2\mathbf{f}}{\partial t^{0,\ell}t^{d,i}}|_{(\mathbf{u}^1,\cdots,\mathbf{u}^s,0,0,\cdots)} \\ &= \eta^{j\ell}\partial_x\frac{\partial^2\mathbf{f}}{\partial t^{0,\ell}t^{d,i}} \\ &= \eta^{j\ell}\frac{\partial^3\mathbf{f}}{\partial t^{0,1}\partial t^{0,\ell}\partial t^{d,i}} \\ &= \frac{\partial\mathbf{u}^j}{\partial t^{di}} \end{aligned}$$

$$
\begin{aligned}
\{\mathbf{h}_{a\alpha}, \mathbf{h}_{b\beta}\}[u] &= \frac{1}{(2\pi)^2} \int_0^{2\pi} \delta_i h_{a\alpha} \eta^{ij} \partial_x \delta_j h_{b\beta} \, \mathrm{d}x \\
&= \frac{1}{2\pi} \int_0^{2\pi} \sum_{s=0}^{\infty} (-\partial_x)^s \frac{\partial h_{a\alpha}}{\partial u^{i,s}}(u(x)) \eta^{ij} \partial_x \sum_{t=0}^{\infty} (-\partial_x)^t \frac{\partial h_{b\beta}}{\partial u^{j,s}}(u(x)) \, \mathrm{d}x \\
\text{($h_{a\alpha}, h_{b\beta}$ do not depend on $x$-derivatives of $u$)} \quad &= \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial^2 \mathbf{f}}{\partial t^{0,i} \partial t^{a\alpha}}(u(x), 0, \cdots) \eta^{ij} \partial_x \frac{\partial^2 \mathbf{f}}{\partial t^{0,j} \partial t^{b\beta}}(u(x), 0, \cdots) \, \mathrm{d}x \\
\text{(by proposition 6.3.1)} \quad &= \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial^2 \mathbf{f}}{\partial t^{0,i} \partial t^{a\alpha}} \eta^{ij} \partial_x \frac{\partial^2 \mathbf{f}}{\partial t^{0,j} \partial t^{b\beta}} \, \mathrm{d}x \\
(\partial_x = \partial_{t^{0,1}}) \quad &= \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial^2 \mathbf{f}}{\partial t^{0,i} \partial t^{a\alpha}} \eta^{ij} \frac{\partial^3 \mathbf{f}}{\partial t^{0,1} \partial t^{0,j} \partial t^{b\beta}} \, \mathrm{d}x \\
\text{(by the TRR)} \quad &= \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial^3 \mathbf{f}}{\partial t^{a+1,\alpha} \partial t^{0,1} \partial t^{b\beta}} \, \mathrm{d}x \\
&= \frac{1}{2\pi} \int_0^{2\pi} \partial_x \Omega_{a+1,\alpha,b,\beta} \, \mathrm{d}x \\
&= 0
\end{aligned}
$$

□

**Example 6.3.4** (The dispersionless Korteweg-de Vries hierarchy)**.** Let $\mathbf{f}$ be the rational Gromov-Witten potential for $M = \{\text{point}\}$. So we have only a single generating cohomology class $e_1$ and a single sequence of variables $t^{0,1}, t^{1,1}, t^{2,1}, \cdots$ for which we will write $t^0, t^1, \cdots$. Then we easily see that $\mathbf{f}_{t^d=0|d>0} = \frac{1}{6}\left(t^0\right)^3$. So we have

$$\mathbf{f}(\mathbf{u}, 0, \cdots) = \frac{1}{6}\mathbf{u}^3$$

and by the equations we derived earlier (note $\eta_{ij} = \delta_{ij}$ and $\eta^{ij} = \delta^{ij}$)

$$
\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t^d} &= \partial_x \Omega_d(\mathbf{u}, 0, \cdots) \\
&= \partial_x \frac{\partial^2 \mathbf{f}}{\partial t^0 \partial t^d}\Big|_{(\mathbf{u},0,\cdots)} \\
\text{(chain rule)} &= \mathbf{u}_x \frac{\partial^3 \mathbf{f}}{\partial t^0 \partial t^0 \partial t^d}\Big|_{(\mathbf{u},0,\cdots)} \qquad (6.3) \\
\text{(for } d = 0) &= \mathbf{u}_x \\
\text{(for } d > 0) \text{ (TRR)} &= \mathbf{u}_x \tfrac{\partial^2 \mathbf{f}}{\partial t^0 \partial t^{d-1}}\Big|_{(\mathbf{u},0,\cdots)} \underbrace{\frac{\partial^3 \mathbf{f}}{\partial t^0 \partial t^0 \partial t^0}\Big|_{(\mathbf{u},0,\cdots)}}_{=1} \\
&= \mathbf{u}_x \frac{\partial^2 \mathbf{f}}{\partial t^0 \partial t^{d-1}}\Big|_{(\mathbf{u},0,\cdots)} \\
\text{(SE)} &= \mathbf{u}_x \frac{\partial}{\partial t^{d-1}} \left( \sum_e t^{e+1} \frac{\partial \mathbf{f}}{\partial t^e} + \left(t^0\right)^2 \right) \Big|_{(\mathbf{u},0,\cdots)} \\
(d = 1) &= \mathbf{u}_x \mathbf{u} \\
(d > 1) &= \mathbf{u}_x \frac{\partial \mathbf{f}}{\partial t^{d-2}}\Big|_{(\mathbf{u},0,\cdots)}
\end{aligned}
$$

Or, just starting from (6.3), we can see that we have in general

$$\frac{\partial \mathbf{u}}{\partial t^d} = \frac{1}{d!} \mathbf{u}^d \mathbf{u}_x .$$

So the Hamiltonians involved are given by

$$
\begin{aligned}
\{u, \mathbf{h}_d\}[\mathbf{u}] &= \frac{1}{d!} \mathbf{u}^d \mathbf{u}_x \\
\partial_x \frac{\partial h_d}{\partial u} &= \frac{1}{d!} u^d u_x \\
h_d &= \frac{1}{(d+1)!} u^{d+1}
\end{aligned}
$$

Let us check that they are in involution.

$$\begin{aligned}
\{h_d, \mathbf{h}_e\} &= \frac{\partial h_d}{\partial u}\partial_x \frac{\partial h_e}{\partial u} \\
&= \frac{1}{(d-1)!}u^{d-1}\partial_x \frac{1}{(e-1)!}u^{e-1} \\
&= \frac{1}{(d-1)!(e-2)!}u^{d+e-3}u_x \\
&= \partial_x \left( \frac{1}{(d+e-2)(d-1)!(e-2)!}u^{d+e-2} \right)
\end{aligned}$$

## 6.4 GROMOV-WITTEN POTENTIAL AND SFT OF CIRCLE BUNDLES

We consider the same situation as in the last section: $(M, \omega)$ is a closed symplectic manifold, with a basis $\Delta_1, \cdots, \Delta_b$ of its cohomology such that $\Delta_1 = 1$, and with Gromov-Witten potential with descendants $\mathbf{f}(t^{d,i}\Delta_i, z)$.

Let $\pi\colon S^1 \times M \to M$ be the trivial $S^1$-bundle and let $\xi$ be the horizontal distribution. Let $\alpha := \frac{\partial}{\partial\phi}$ where $\mathrm{e}^{2\pi i\phi} \mapsto \phi$ is a coordinate along $S^1$. Then $(S^1 \times M, \xi)$ becomes a contact manifold with contact form $\alpha$, and the fibres of $\pi$ correspond to Reeb orbits. We can find a basis for its cohomology of the form $\tilde{\Delta}_1, \cdots, \tilde{\Delta}_b, \tilde{\Theta}_1, \cdots, \tilde{\Theta}_c$, where the $\tilde{\Delta}_i$'s are pullbacks $\pi^*\Delta_i$. This gives us a rational SFT-Hamiltonian (with descendants) $\mathbf{h}(\sum_{d,i} t_{d,i}\tilde{\Delta}_i + \sum_{d,i} \tau_{d,i}\tilde{\Theta}_i, q, p, z)$. We would like to find out how it is related to the Gromov-Witten potential.

The relation we describe will, on the side of the Gromov-Witten potential, be in terms of the densities of the Hamiltonians that we studied in the last section:

$$h_{d,i}(u) = \frac{\partial \mathbf{f}}{\partial t^{d,i}}|_{(u^1,\cdots,u^s,0,0,\cdots)}\,.$$

On the side of the SFT-Hamiltonian, it will be in terms of the functionals

$$\mathbf{h}_{d,i}[u] = \frac{\partial \mathbf{h}}{\partial \tau^{d,i}}(u_0^0[u], \cdots, u_0^s[u], 0, 0\cdots; 0, \cdots, 0; q[u]; p[u])$$

(Remember that the Fourier coefficients $u_0^j, q_k^j, p_k^j$ are local functionals given by

$$\begin{aligned} u_0^j[u] &= \frac{1}{2\pi}\int_0^{2\pi} u^j \,\mathrm{d}x \\ q_k^j[u] &= \frac{1}{2\pi}\int_0^{2\pi} u^j \mathrm{e}^{+\mathrm{i}kx}\,\mathrm{d}x \\ p_k^j[u] &= \frac{1}{2\pi}\int_0^{2\pi} u^j \mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x \qquad ). \end{aligned}$$

Note that it is not immediately obvious that $\mathbf{h}_{d,i}$ is local. However, we have the following

**Theorem 6.4.1.** *With the above definitions, we have*

$$\mathbf{h}_{d,i}[u] = \frac{1}{2\pi}\int_0^{2\pi} h_{d,i}(u)\,\mathrm{d}x$$

*Proof.* Let us abbreviate the formulae by writing $p_0$ for $u_0$, and $p_{-n}$ for $q_n$, and by writing $s$ for $r + s^- + s^+$. Then:

$$\mathbf{h}_{d,i}[u] = \sum_{r,s^-,s^+}\sum_{j_1,\cdots,j_s}\frac{1}{r!s^-!s^+!}\int_{\overline{\mathcal{M}}(V\times\mathbf{R})/\mathbf{R}} \cdots \mathrm{ev}^*\left(\tilde{\Delta}_i \wedge \mathrm{d}\phi\right)\psi_i^d p_{k_1}^{j_1}\cdots p_{k_s}^{j_s} \tag{6.4}$$

and

$$h_{d,i}(u) = \sum_r\sum_{i_1,\cdots,i_r}\frac{1}{r!}\int_{\overline{\mathcal{M}}_r(M)} \cdots \mathrm{ev}^*(\Delta_i\psi_i^d)u^{i_1}\cdots u^{i_s}$$

We will substitute the Fourier expansion of $u$ , giving

$$h_{d,i}(u) = \sum_r\sum_{i_1,\cdots,i_r}\frac{1}{r!}\int_{\overline{\mathcal{M}}(M)} \mathrm{ev}^*(\cdots\Delta_i\psi_i^d)(\sum_{k_1} p_{k_1}^{i_1}\mathrm{e}^{\mathrm{i}k_1x})\cdots(\sum_{k_s} p_{k_s}^{i_r}\mathrm{e}^{\mathrm{i}k_sx})$$

So after integrating over $x$, we get

$$\frac{1}{2\pi}\int_0^{2\pi} h_{d,i}(u)\,\mathrm{d}x = \sum_r \sum_{\substack{i_1,\cdots,i_r \\ 0=k_1+\cdots+k_r}} \frac{1}{r!}\int_{\overline{\mathcal{M}}} \mathrm{ev}^*(\cdots\Delta_i\psi_i^d)p_{k_1}^{i_1}\cdots p_{k_r}^{i_r} \tag{6.5}$$

Comparing (6.4) and (6.5), we see that we should have

$$\frac{1}{r!s^-!s^+!} \int_{\overline{\mathcal{M}}_r(s^-,s^+;V\times\mathbf{R})/\mathbf{R}} \mathrm{ev}^* \cdots \left(\tilde{\Delta}_i \wedge \mathrm{d}\phi\right) \psi_i^d = \begin{cases} \frac{1}{s!} \int_{\overline{\mathcal{M}}_{r+s^-+s^+}(M)} \mathrm{ev}^*(\cdots \Delta_i \psi_i^d) & \text{if } k_1 + \cdots + k_s = 0 \\ 0 & \text{otherwise} \end{cases}$$

So let us explore the relation between the moduli spaces

$$\overline{\mathcal{M}}_{0,r}(s^-,s^+;V\times\mathbf{R})/\mathbf{R}$$

and

$$\overline{\mathcal{M}}_{0,r+s^-+s^+}(M)\,.$$

First of all, we see that $V\times\mathbf{R}$ can be seen as a $\mathbf{C}^*$-bundle over $M$, and, since the radial component is a trivial bundle, we can extend it to a complex line bundle $L\to M$. Let $\tilde{C} = (\tilde{C}, x^{\pm}, y, f)$ be a 1-story curve in $\overline{\mathcal{M}}_{0,r}(s^-,s^+;V\times\mathbf{R})$. Then projecting down gives a curve $C$ in $\overline{\mathcal{M}}_{0,r+s^-+s^+}(M)$. We can regard $\tilde{C}$ as a section $s$ of $L$ over $C$. This section will have zeroes at the points $\pi(x^-)$ and poles at the points $\pi(x^+)$, with multiplicities corresponding to the multiplicities of the orbits in $\Gamma^{\pm}$. We get a divisor $[s]$ on $C$, with support in the last $s^-+s^+$ marked points on $C$. Since $L$ is a trivial bundle, $\deg[s]=0$.

Conversely, given such a divisor $[s]$ of degree 0, it determines the section $s$, and therefore an element $\tilde{C}$ of $\overline{\mathcal{M}}_{0,r}(s^-,s^+;W\times\mathbf{R})$, up to a multiplicative constant in $\mathbf{C}^*$. After modding out by the $\mathbf{R}$-action, we find that it determines an $S^1$-bundle of curves in $\overline{\mathcal{M}}_{0,r}(s^-,s^+;V\times\mathbf{R})/\mathbf{R}$. So we get the diagram (writing $\mathrm{div}_0(\overline{\mathcal{M}}_{r+s^-+s^+}(M))$ for the bundle whose fibre over $C$ are the degree 0 divisors with support in the last $s^-+s^+$ marked points)

$$\underset{\circlearrowright\, S^1}{\overline{\mathcal{M}}(V\times\mathbf{R})} \twoheadrightarrow \mathrm{div}_0(\overline{\mathcal{M}}(M)) \twoheadrightarrow \overline{\mathcal{M}}(M)$$

These maps respect pullbacks of the $\Delta_i$'s, and $\mathrm{d}\phi$ integrates to 1 over the $S^1$-symmetry. The right hand map has discrete fibres, corresponding to the divisors with support in the last $s^-+s^+$ marked points. This corresponds to the covering number of the Reeb orbits over which $\tilde{C}$ is asymptotically cylindrical. The combinatorial factor $\binom{r+s^-+s^+}{r;s^-;s^+}$ is explained by the fact that $\overline{\mathcal{M}}_{r+s^-+s^+}(M)$ does not distinguish between the marked points and the punctures. The theorem follows. □

A slight extension of the above argument also applies to non-trivial circle bundles. We will illustrate this in the following example.

**Example 6.4.2** (Toda lattice)**.** Let $M = \mathbf{C}P^1$ and $V = S^3$, which we regard as the Hopf-fibration $\pi\colon V \to M$. Let $W = V \times \mathbf{R}$. We can regard $W$ as the total space of the line bundle $\mathcal{O}(-1)$ over $M$, minus the zero section. We write $H^\bullet(M) = \langle \theta_0 = 1, \theta_2 \rangle$ with $\int_M \theta_2 = 1$; so $\eta_{ij} = \eta^{ij} = (1 - \delta_{ij})$.

We can relate the SFT of $V$ to the G-W of $M$. For this, recall that $\pi$ induces a map $\pi_*\colon \overline{\mathcal{M}}^A_{0,r}(s^-, s^+; W) \to \overline{\mathcal{M}}^{\pi_* A}_{0,r+s^-+s^+}(M)$. Let us write $\overset{\vee}{f} := \pi_* f$ for $f$ in the SFT-modspace, and let $d = \int_{\pi_* A} \theta_2$ be its degree. We can regard $f$ as a section of the line bundle $\overset{\vee}{f}{}^* \mathcal{O}(-1) = \mathcal{O}(-d)$, with poles at the marked points $x_1^+ \cdots x_{s^+}^+$ and zeroes at $x_1^- \cdots x_{s^-}^-$. Suppose that these marked points have multiplicities $\kappa_1^+ \cdots \kappa_{s^+}^+$ and $\kappa_1^- \cdots \kappa_{s^-}^-$; then we see that

$$\sum \kappa_i^- - \sum \kappa_j^+ = d \tag{6.6}$$

Let us now substitute

$$\begin{aligned} t^{0,i} &\mapsto u^i := u_0^i + \sum (p_\kappa^i \mathrm{e}^{\mathrm{i}\kappa x} + q_\kappa^i \mathrm{e}^{-\mathrm{i}\kappa x}) \\ z_2 &\mapsto \mathrm{e}^{-\mathrm{i}x} \end{aligned}$$

into $\mathbf{f}$, and integrate over $x$. Then the only nonzero contributions come from monomials in $\{p_\kappa^i, q_\kappa^i\}$ that satisfy (6.6); this then exactly gives

$$\mathbf{h}_{d,i} := \frac{\partial \mathbf{h}}{\partial \tau^{d,i}}\Big|_{(\tau=0, t^{0,i}=u^i, t^{d,i}=0)} = \int_{S^1} \frac{\partial \mathbf{f}}{\partial t^{d,i}} (u^1 \cdots u^s, 0, \cdots)\, \mathrm{d}x$$

This allows us to calculate $\mathbf{h}$. We know that without descendants, $\mathbf{f} = \frac{1}{2} t_1^2 t_2 + \mathrm{e}^{t_2} z_2$. Then the above gives

$$\begin{aligned} \mathbf{h}_{0,1}[u] &= \int_{S^1} u^1 u^2 \, \mathrm{d}x \\ \mathbf{h}_{0,2}[u] &= \int_{S^1} \frac{1}{2} \left(u^1\right)^2 + \mathrm{e}^{u^2 - \mathrm{i}x} \, \mathrm{d}x \end{aligned}$$

giving the equations of motion

$$\begin{aligned}
\frac{\partial u^1}{\partial t^{0,1}} &= \{u^1, \mathbf{h}_{0,1}\} \\
\text{(remember that } \eta^{ij} = 1 - \delta^{ij}\text{)} &= \partial_x \frac{\partial h_{0,1}}{\partial u^2}(u^1, u^2) \\
&= \partial_x u^1 \qquad \text{(check)} \\
\frac{\partial u^2}{\partial t^{0,1}} &= \partial_x \frac{\partial h_{0,1}}{\partial u^1}(u^1, u^2) \\
&= \partial_x u^2 \qquad \text{(check)} \\
\frac{\partial u^1}{\partial t^{0,2}} &= \{u^1, \mathbf{h}_{0,2}\} \\
&= \partial_x \frac{\partial h_{0,2}}{\partial u^2}(u^1, u^2) \\
&= \partial_x \left(\mathrm{e}^{u^2 - \mathrm{i}x}\right) \\
&= \left(u^2_x - \mathrm{i}\right) \mathrm{e}^{u^2 - \mathrm{i}x} \\
\frac{\partial u^2}{\partial t^{0,2}} &= \partial_x \frac{\partial h_{0,2}}{\partial u^1}(u^1, u^2) \\
&= \partial_x u^1 = u^1_x \\
\text{(taking these together)} \ \frac{\partial^2 u^2}{\partial \left(t^{0,2}\right)^2} &= (\partial_x)^2 \left(\mathrm{e}^{u^2 - \mathrm{i}x}\right)
\end{aligned}$$

Note that this in not an integrable hierarchy on the loop space. This is explained by the substitution $z \mapsto \mathrm{e}^{-\mathrm{i}x}$.

## 6.5 THE DUBROVIN-ZHANG PRINCIPAL HIERARCHY – TAKE II

We will now describe the same hierarchy on the same space $\mathcal{L}(H^\bullet(M))$, but from the different perspective of symplectic field theory. Symplectic field theory gives new proofs of the structure of the integrable system in the Gromov-Witten case. This has been previously done in [11].

We start with the same bundle $\pi\colon S^1 \times M \to M$ as in the last section. It is a framed Hamiltonian structure. We write $\mathbf{h}(t;\tau;q;p)$ for its rational SFT Hamiltonian with descendants, as defined in section 4.5. Here, the variables $t^{d,i}$ are associated to the basis $\Delta_1, \cdots, \Delta_s$ of $H^\bullet(M)$, and the variables $\tau^{d,i}$ are associated to $\Delta_i \wedge \mathrm{d}\phi$ for a coordinate $\phi$ on $S^1$. We let

$$\mathbf{h}_{d,i}[u] := \frac{\partial \mathbf{h}}{\partial \tau^{d,i}}(t^{0,1}, \cdots, t^{0,s}, 0, 0, \cdots; 0, \cdots, 0; q; p)$$

where we have expressed $u \in \mathcal{L}$ as

$$u^j = t^{0,j} + \sum_{k=1}^{\infty} q^j_k e^{-ikx} + p^j_k e^{+ikx}$$

By theorem 6.4.1, $\mathbf{h}_{d,i}$ is indeed a local functional, with density

$$h_{d,i} = \frac{\partial \mathbf{f}}{\partial t^{d,i}}|_{(u^1,\cdots,u^s,0,0\cdots)}$$

By proposition 5.3.1, we have $\{\mathbf{h}, \mathbf{h}\} = 0$. We can expand $\mathbf{h}$ in the number of $\tau$ variables as

$$\mathbf{h} = \mathbf{h}_0(t;q;p) + \mathbf{h}_{d,i}(t;q;p)\tau^{d,i} + \mathbf{h}_{d_1 i_1 d_2 i_2}(t;q;p)\tau^{d_1,i_1}\tau^{d_2,i_2} + \cdots$$

Note that $\mathbf{h}_{d,i}$ agrees with our previous definition. We get

$$\begin{aligned}
0 &= \{\mathbf{h}, \mathbf{h}\} \\
&= \{\mathbf{h}_0, \mathbf{h}_0\} + \{\mathbf{h}_0, \mathbf{h}_{d,i}\tau^{d,i}\} + \{\mathbf{h}_{d,i}\tau^{d,i}, \mathbf{h}_0\} + \{\mathbf{h}_{d_1,i_1}\tau^{d_1,i_1}, \mathbf{h}_{d_2,i_2}\tau^{d_2,i_2}\} + \cdots \\
&= \{\mathbf{h}_0, \mathbf{h}_0\} \\
&\quad + \{\mathbf{h}_0, \mathbf{h}_{d,i}\}\tau^{d,i} + \mathbf{h}_{d,i}\{\mathbf{h}_0, \tau^{d,i}\} + \mathbf{h}_{d,i}\{\tau^{d,i}, \mathbf{h}_0\} + \{\mathbf{h}_{d,i}, \mathbf{h}_0\}\tau^{d,i} \\
&\quad + \mathbf{h}_{d_1,i_1}\mathbf{h}_{d_2,i_2}\{\tau^{d_1,i_1}, \tau^{d_2,i_2}\} + \mathbf{h}_{d_2,i_2}\{\mathbf{h}_{d_1,i_1}, \tau^{d_2,i_2}\}\tau^{d_1,i_1} \\
&\qquad + \mathbf{h}_{d_1,i_1}\{\tau^{d_1,i_1}, \mathbf{h}_{d_2,i_2}\}\tau^{d_2,i_2} + \{\mathbf{h}_{d_1,i_1}, \mathbf{h}_{d_2,i_2}\}\tau^{d_1,i_1}\tau^{d_2,i_2} + \cdots \\
&= \{\mathbf{h}_0, \mathbf{h}_0\} \\
&\quad + 2\{\mathbf{h}_0, \mathbf{h}_{d,i}\}\tau^{d,i} \\
&\quad + \left(\{\mathbf{h}_{d_1,i_1}, \mathbf{h}_{d_2,i_2}\} + 2\{\mathbf{h}_0, \mathbf{h}_{d_1 i_1 d_2 i_2}\}\right)\tau^{d_1,i_1}\tau^{d_2,i_2} + \cdots
\end{aligned}$$

Since the coefficients for all $\tau$-monomials must be zero, we get

$$\begin{aligned}
\{\mathbf{h}_0, \mathbf{h}_0\} &= 0 \\
\{\mathbf{h}_0, \mathbf{h}_{d,i}\} &= 0 \\
\{\mathbf{h}_{d,i}, \mathbf{h}_{e,j}\} &= -2\{\mathbf{h}_0, \mathbf{h}_{d,i,e,j}\}
\end{aligned}$$

The moduli spaces inherit the $S^1$-symmetry of the bundle, and the $t$-variables correspond to the forms $\pi^*\Delta$ which have no component in this direction. We conclude that $\mathbf{h}_0 = 0$. Then the first two equations do not tell us much. However, the last one gives

$$\{\mathbf{h}_{d,i}, \mathbf{h}_{e,j}\} = 0\,.$$

So we have found a commuting system of Hamiltonians.

Let us see what this tells us in terms of the densities $h_{d,i}, h_{e,j}$. Let us define $\Omega_{d,i,e,j}$ as in (6.1), so

$$\partial_x \Omega_{d+1,i,e,j} = \{h_{d,i}, \mathbf{h}_{e,j}\}$$

Then applying lemma 6.3.2 (a consequence of the string equation) we get

$$\begin{aligned}
\partial_x \Omega_{d,i,e,j} &= \frac{\partial h_{d-1,i}}{\partial u^i} \eta^{ij} \frac{\partial^2 h_{e,j}}{\partial u^j \partial u^1} \\
& \frac{\partial^2 h_{d,i}}{\partial u^1 \partial u^i} \eta^{ij} \frac{\partial^2 h_{e,j}}{\partial u^j \partial u^1} \\
& \frac{\partial^2 h_{d,i}}{\partial u^1 \partial u^i} \eta^{ij} \frac{\partial h_{e+1,j}}{\partial u^j} \\
&= \partial_x \Omega_{e,j,d,i}
\end{aligned}$$

So $\Omega_{d,i,e,j} = \Omega_{e,j,d,i}$. This means that the system of Hamiltonians $\mathbf{h}_{d,i}$ is *tau-symmetric.*

By (6.2), we have

$$\frac{\partial^2 h_{d,i}}{\partial u^m \partial u^k} \eta^{k\ell} \frac{\partial^2 h_{e,j}}{\partial u^\ell \partial u^n} = \frac{\partial^2 h_{d,i}}{\partial u^n \partial u^k} \eta^{k\ell} \frac{\partial^2 h_{e,j}}{\partial u^\ell \partial u^m}$$

In terms of the Gromov-Witten potential, we get

$$\frac{\partial^3 \mathbf{f}}{\partial t^{0,m} \partial t^{d,i} \partial t^{0,k}} \eta^{k\ell} \frac{\partial^3 \mathbf{f}}{\partial t^{0,\ell} \partial t^{e,j} \partial t^{0,n}} = \frac{\partial^3 \mathbf{f}}{\partial t^{0,n} \partial t^{d,i} \partial t^{0,k}} \eta^{k\ell} \frac{\partial^3 \mathbf{f}}{\partial t^{0,\ell} \partial t^{e,j} \partial t^{0,m}}$$

where we must substitute $t^{a\alpha} \mapsto 0$ for $a > 0$. In the case $d = e = 0$, we have derived the *Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equation* for the Gromov-Witten potential (without descendants).

Now consider the case $e = 0$ and $n = 1$. By the string equation, we have

$$\frac{\partial^3 \mathbf{f}}{\partial t^{0,\ell} \partial t^{0,j} \partial t^{0,1}} = \eta_{ij}$$

so we get

$$\begin{aligned}
\frac{\partial^3 \mathbf{f}}{\partial t^{0,m} \partial t^{d,i} \partial t^{0,k}} \eta^{k\ell} \eta_{\ell j} &= \frac{\partial^3 \mathbf{f}}{\partial t^{0,1} \partial t^{d,i} \partial t^{0,k}} \eta^{k\ell} \frac{\partial^3 \mathbf{f}}{\partial t^{0,\ell} \partial t^{0,j} \partial t^{0,m}} \\
\implies \frac{\partial^3 \mathbf{f}}{\partial t^{0,m} \partial t^{d,i} \partial t^{0,j}} &= \frac{\partial^3 \mathbf{f}}{\partial t^{0,1} \partial t^{d,i} \partial t^{0,k}} \eta^{k\ell} \frac{\partial^3 \mathbf{f}}{\partial t^{0,\ell} \partial t^{0,j} \partial t^{0,m}}
\end{aligned}$$

and by lemma 6.3.2 we get

$$\frac{\partial^3 \mathbf{f}}{\partial t^{0,m} \partial t^{d,i} \partial t^{0,j}} = \frac{\partial^2 \mathbf{f}}{\partial t^{d-1,i} \partial t^{0,k}} \eta^{k\ell} \frac{\partial^3 \mathbf{f}}{\partial t^{0,\ell} \partial t^{0,j} \partial t^{0,m}}$$

(where we must again substitute $t^{a\alpha} \mapsto 0$ for $a > 0$) which is a special case of the *topological recursion relation* for the Gromov Witten potential.

# BIBLIOGRAPHY

[1] F. Bourgeois, Y. Eliashberg, H. Hofer, K. Wysocki, and E. Zehnder, *Compactness results in symplectic field theory*, Geometry and Topology **7** (2003), no. 2, 799–888.

[2] Y. Eliashberg, A. Givental, and H. Hofer, *Introduction to symplectic field theory*, Geom. Funct. Anal (2000), 560–673.

[3] O. Fabert, *Gravitational Descendants in Symplectic Field Theory*, ArXiv preprint `arXiv:0907.0789` (2009).

[4] O. Fabert and P. Rossi, *String, dilaton and divisor equation in Symplectic Field Theory*, ArXiv preprint `arXiv:1001.3094` (2010).

[5] ———, *Topological recursion relations in non-equivariant cylindrical contact homology*, ArXiv preprint `arXiv:1007.2287` (2010).

[6] A. Floer, H. Hofer, and D. Salamon, *Transversality in elliptic Morse theory for the symplectic action*, Duke Mathematical Journal **80** (1995), no. 1, 251–292.

[7] M. Hutchings, *Lecture notes on Morse homology*, 2002.

[8] D. McDuff and D. Salamon, *J-holomorphic curves and quantum cohomology*, Amer Mathematical Society, 1994.

[9] ———, *Introduction to Symplectic Topology*, Oxford Science Publications, 1998.

[10] ———, *J-holomorphic curves and symplectic topology*, Amer Mathematical Society, 2004.

[11] P. Rossi, *Integrable systems and holomorphic curves*, ArXiv preprint `arXiv:0912.0451` (2009).

[12] D. Salamon, *Morse theory, the Conley index and Floer homology*, Bulletin of the London Mathematical Society **22** (1990), no. 2, 113.

[13] ———, *Lectures on Floer homology*, IAS/Park City Mathematics Series **7** (1999), 145–229.

[14] E. Witten, *On the structure of the topological phase of two-dimensional gravity*, Nuclear Physics B **340** (1990), no. 2-3, 281–332.

# INDEX