Calogero-Moser system on an elliptic curve

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Main point

We will see a *complicated* system of interacting particles that can be solved because it corresponds to a *simple* motion in the space of holomorphic principal $\mathrm{SL}(N)$-bundles over an elliptic curve.

Outline

Classical integrability

Marsden-Weinstein reduction

Reduction for the elliptic case

Phase space

Definition

A *phase space* or a *symplectic manifold* is a manifold M together with a non-degenerate, closed 2-form ω.

Definition

For any function $f \in C^\infty(M)$ on a symplectic manifold, there is an associated vector field v_f given by

$$\omega^{-1}(\mathrm{d}f)$$

where we regard ω as a bundle map $TM \to T^\vee M$. We also write

$$\{f, \cdot\}$$

for this vector field.

Phase space (ctd)

Example

Any cotangent bundle $M = T^\vee X$ is a symplectic manifold.

- $q_1, \cdots, q_n$ and $p^1, \cdots, p^n$ coordinates representing a point $\left(p^i \mathrm{d}q_i, q\right)$
- Symplectic form:

$$\begin{aligned} \{q_i, \cdot\} &= -\frac{\partial}{\partial p^i} \\ \{p^i, \cdot\} &= \frac{\partial}{\partial q_i} \end{aligned}$$

Liouville integrability

Definition

A *dynamical system* is a phase space together with a distinguished function $H \in C^\infty(M)$ called the *Hamiltonian*. The *solutions* to the dynamical system are the flow lines of $\{H, \cdot\}$.

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Definition

Let $\dim M = 2N$. A dynamical system on M is *(Liouville) integrable* if there are functions $H_1, \cdots, H_N$ such that

- $\{H_i, H_j\} = 0$ (they are *in involution*)
- On a dense open subset: $\mathrm{d}H_1 \wedge \cdots \wedge \mathrm{d}H_N \neq 0$
- $H = f(H_1, \cdots, H_N)$

Liouville integrability (ctd)

The H_i are called *Hamiltonians.* Their flows are symmetries of the dynamical system.

Planetary motion

Example

$M = T^\vee\left(\mathbf{R}^3 \times \mathbf{R}^3\right)$, with coordinates $x, y \in \mathbf{R}^3$ and cotangent coordinates p, r. We interpret x, y as planet positions and p, r as momenta.

$$\begin{aligned} \omega &= \mathrm{d}p_i \wedge \mathrm{d}x^i + \mathrm{d}r_i \wedge \mathrm{d}y^i \\ H &= \frac{1}{2}\left(p^2 + q^2\right) + \frac{1}{|x-y|} \end{aligned}$$

This is integrable with

$$\begin{aligned} H_k &= p_k + r_k \qquad k = 1,2,3 \\ H_4 &= ((p-r) \times (x-y))_1 \\ H_5 &= \left|(p-r) \times (x-y)\right|^2 \\ H_6 &= H \end{aligned}$$

Outline

Rational Calogero-Moser system

Example

$M \subseteq T^\vee \mathbf{R}^N$, interpreted as positions and momenta of N particles in 1 dimension with center-of-mass set to 0.

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 - \sum_{i<j} \frac{\epsilon^2}{(q_i - q_j)^2}$$

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Theorem (Calogero)

This is an $2(N-1)$-dimensional integrable system, with Hamiltonians given by $H_k = \operatorname{Tr} L^k$, where L is the traceless matrix

$$L = \begin{pmatrix} p_1 & & \frac{\epsilon}{q_i - q_j} \\ & \ddots & \\ \frac{\epsilon}{q_i - q_j} & & p_N \end{pmatrix}$$

Rational Calogero-Moser system (ctd)

So some of these Hamiltonians are:

$$H_1 = \sum_{i=1}^{N} p_i = 0$$

$$H_2 = \sum_{i=1}^{N} p_i^2 - \sum_{i \neq j} \frac{\epsilon^2}{(q_i - q_j)^2} \qquad (= 2H)$$

$$H_3 = \sum_{i=1}^{N} p_i^3 - \sum_{i \neq j} p_i \frac{\epsilon^2}{(q_i - q_j)^2} + \sum_{i,j,k \text{ distinct}} \frac{\epsilon^3}{(q_i - q_j)(q_j - q_k)(q_k - q_i)}$$

Rational Calogero-Moser system (ctd)

Question

Where do all these symmetries / conserved quantities come from?

Rational Calogero-Moser system (ctd)

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"Answer"

They exist because the motion is very simple (linear) in the matrix space.

Linear motion in a matrix space

- $G = \mathrm{SL}(N)$, $\mathfrak{g} = \mathfrak{sl}(N)$
- Phase space $M = T^\vee \mathfrak{g} = \mathfrak{g} \times \mathfrak{g}$ using Killing pairing
- Hamiltonian $H(P, Q) = \frac{1}{2} \langle P, P \rangle$

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- Hamiltonian $H(P, Q) = \frac{1}{2}\langle P, P\rangle$
- Solution for given initial value (P_0, Q_0):

$$\begin{aligned} P(t) &= P_0 \\ Q(t) &= Q_0 + tP_0 \end{aligned}$$

Linear motion in a matrix space (ctd)

- Symmetric under adjoint action of G on $\mathfrak{g}$:
  - H and ω invariant under conjugation $P, Q \mapsto gPg^{-1}, gQg^{-1}$
  - Time evolution commutes with G-action
- Conserved quantities in involution:

$$H_k = \mathrm{Tr}\, P^k$$

- Also invariant under conjugation
- But too few: $\dim M = 2\left(n^2 - 1\right) > 2n$

Marsden-Weinstein reduction

Idea

- Since everything is G-invariant, we can quotient out by it.
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- Since everything is G-invariant, we can quotient out by it.
- Hopefully, this reduces the dimension sufficiently to end up with an integrable system.
- But we also need to keep a non-degenerate symplectic form:
  - If we quotient out a tangent vector $\xi \in TM$, then we should also quotient out its image $\omega(\xi) \in T^\vee M$. Dually, that means restricting to a submanifold.

Marsden-Weinstein reduction (ctd)

Definition

A group action of G on M is *Hamiltonian* if its infinitesimal vector fields v_ξ for $\xi \in \mathfrak{g}$ are of the form

$$v_\xi = \{f_\xi, \cdot\}$$

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Definition

A Hamiltonian group action is *generated by a moment map* $\mu\colon M \to \mathfrak{g}^\vee$ if

$$f_\xi = \langle \mu, \xi \rangle$$

and if μ is G-equivariant.

Marsden-Weinstein reduction (ctd)

Theorem (Marsden, Weinstein)

If $\mu_0 \in \mathfrak{g}^\vee$ is a regular value of μ, then the space $\mu^{-1}(\mu_0)/G_{\mu_0}$ is a symplectic manifold.

Reduction of linear motion

Fact

$\mu(P,Q) = [P,Q] \in \mathfrak{g} \cong \mathfrak{g}^\vee$ is a moment map for the conjugation action.

Theorem

Pick the following regular value $\mu_0 \in \mathfrak{g} \cong \mathfrak{g}^\vee$:

$$\mu_0 = -\epsilon \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} + \epsilon \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

Then p, q parametrize the G_{μ_0}-equivalence classes in $\mu^{-1}(\mu_0)$ by

$$P(p,q), Q(p,q) = \begin{pmatrix} p_1 & & \frac{\epsilon}{q_i - q_j} \\ & \ddots & \\ \frac{\epsilon}{q_i - q_j} & & p_N \end{pmatrix}, \begin{pmatrix} q_1 & & 0 \\ & \ddots & \\ 0 & & q_N \end{pmatrix}$$

and they are canonical coordinates on $\mu^{-1}(\mu_0)/G_{\mu_0}$.

Recap

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$$t \mapsto P_0, Q_0 + tP_0$$

has $N - 1$ conserved quantities in an $2(N^2 - 1)$ dimensional phase space.

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Recap

- Linear motion
$$t \mapsto P_0, Q_0 + tP_0$$
has $N-1$ conserved quantities in an $2(N^2-1)$ dimensional phase space.
- Is is also symmetric under conjugation
$P, Q \mapsto gPg^{-1}, gQg^{-1}$
- Quotienting out and restricting yields a $2(N-1)$ dimensional space: the Calogero-Moser integrable system

Outline

Particles on an elliptic curve

- (Σ, p) elliptic curve with Weierstrass function $\wp\colon \Sigma \to \mathbf{C}P^1$
- $q = (q_1, \cdots, q_N) \in \Sigma^N$ positions of N particles on the curve
- Complex phase space $T^\vee \Sigma^N$ with coordinates p, q.
- Hamiltonian:

$$H(p,q) = \frac{1}{2}\sum_{i=1}^{N} p_i^2 - \epsilon^2 \sum_{i<j} \wp(q_i - q_j)$$

Large phase space

- $G = \mathrm{SL}(N, \mathbf{C})$; $\mathfrak{g} = \mathfrak{sl}(N, \mathbf{C})$
- $G^\Sigma = C^\infty(\Sigma, G)$, $\mathfrak{g}^\Sigma = C^\infty(\Sigma, \mathfrak{g})$
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- Complex phase space $T^\vee \mathfrak{g}^\Sigma$
- Pick universal cover $\mathbf{C} \to \Sigma$ with coordinate $z \in \mathbf{C}$ such that $0 \mapsto p$
- G^Σ-action:

$$\begin{aligned} Q(z,\bar{z}) &\mapsto g(z,\bar{z})Q(z,\bar{z})g^{-1}(z,\bar{z}) - \partial_{\bar{z}} g g^{-1} \\ P(z,\bar{z}) &\mapsto g(z,\bar{z})P(z,\bar{z})g(z,\bar{z})^{-1} \end{aligned}$$

(*centrally extended co-adjoint action* on Q, or *gauge transformations* on Q)

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  (*centrally extended co-adjoint action* on Q, or *gauge transformations* on Q)
- Hamiltonian action with moment map:

$$\mu(Q, P) = [Q, P] - \partial_{\bar{z}} P$$

Reduction

- Conserved quantities:

$$H_k = \int_\Sigma \operatorname{Tr} P^k \mathrm{d}z\mathrm{d}\bar{z}$$

$$H = \frac{1}{2} H_2$$

- Pick the following value $\mu_0 \in \left(\mathfrak{g}^\Sigma\right)^\vee$:

$$\mu_0 = \left(\epsilon \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} - \epsilon \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}\right) \delta(z, \bar{z})$$

$$=: \tau_0 \delta(z, \bar{z})$$

- Then $\mu^{-1}(\mu_0)/G_{\mu_0}$ is a symplectic manifold

Solving moment map constraint

- Want to parametrize equivalence classes in $\mu^{-1}(\mu_0)/G_{\mu_0}$:

$$\begin{aligned} \mu(Q,P) &= \mu_0 \\ \Longleftrightarrow \qquad [Q,P] - \partial_{\bar{z}} P &= \tau_0 \delta(z,\bar{z}) \end{aligned}$$

1. Diagonalize Q by a gauge transformation to a *constant* diagonal matrix:

$$\begin{aligned} Q &= g\,\mathrm{diag}(q_1,\cdots,q_N)g^{-1} - \partial_{\bar{z}} g g^{-1} \\ &= \mathrm{diag}(q_1,\cdots,q_N)^g \end{aligned}$$

2. g is unique if we require $g(z=0) \in G_{\tau_0}$
3. Then we find

$$\left(\partial_{\bar{z}} - (q_i - q_j)\right) P^g_{ij} = -(\tau_0)_{ij}\,\delta(z,\bar{z})$$

which has a unique solution for P^g_{ij}.

Intermezzo

Line bundles on an elliptic curve

Intermezzo

Line bundles on an elliptic curve

Solution to

$$(\partial_{\bar{z}} + s)\,\Phi = \tau\delta(z,\bar{z})$$

given in terms of Weierstrass σ and ζ function by

$$\begin{aligned}
\Phi(z,s) &= \tau\frac{\sigma(z-\nu s)}{\sigma(z)\sigma(\nu s)}\exp(\alpha s z - s\bar{z}) \\
\nu &= \frac{2}{\pi \mathrm{i}}(\omega_1\bar{\omega}_2 - \bar{\omega}_1\omega_2) \\
\alpha &= \frac{1}{\omega_1}\left(\bar{\omega}_1 + \nu\zeta(\omega_1)\right).
\end{aligned}$$

where ω_1 and ω_2 are defined by

$$\ker(\mathbf{C} \to \Sigma) = 2\omega_1\mathbf{Z} + 2\omega_2\mathbf{Z}$$

Diagonalizing by gauge transformations

- Similarly: sheaf of solutions to

$$\left(\bar{\partial} + \xi(z, \bar{z})\right) h = 0$$

  ($h \in G$) forms a principal G-bundle P_ξ
- Isomorphic iff there exists $g \in G^\Sigma$

$$g \left(\bar{\partial} + \xi(z, \bar{z})\right) g^{-1} = \bar{\partial} + \eta(z, \bar{z})$$

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- Assume $2\omega_1 = 1$. Pull $P_\xi \to \Sigma$ back to the cylinder $\hat{P}_\xi \to \mathbf{C}/\mathbf{Z}$. The pullback $\hat{P}_\xi$ is trivial as a holomorphic bundle (G connected)

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- Assume $2\omega_1 = 1$. Pull $P_\xi \to \Sigma$ back to the cylinder $\hat{P}_\xi \to \mathbf{C}/\mathbf{Z}$. The pullback $\hat{P}_\xi$ is trivial as a holomorphic bundle (G connected)
- Then the holonomy along $2\omega_2 \in \mathbf{C}/\mathbf{Z}$ is almost always conjugate to $\exp(2\bar{\omega}_2\eta)$ with η diagonal. Then $P_\xi \cong P_\eta$

Solution to moment map constraint

Thanks to intermezzos: we can diagonalize Q and we can solve for P_{ij}.

Solution to moment map constraint

Thanks to intermezzos: we can diagonalize Q and we can solve for P_{ij}.
Therefore, p, q parametrize the G_{μ_0}-equivalence classes in $\mu^{-1}(\mu_0)$ by

$$Q = \begin{pmatrix} q_1 & & \\ & \ddots & \\ & & q_N \end{pmatrix}$$

$$P = \begin{pmatrix} p_1 & & \epsilon\Phi(z, q_j - q_i) \\ & \ddots & \\ \epsilon\Phi(z, q_j - q_i) & & p_N \end{pmatrix}$$

and we can check that p, q are canonical coordinates.

Hamiltonian

Pick coordinate z such that $\nu = 1$. Then the Hamiltonian on the reduced space is

$$
\begin{aligned}
H &= \frac{1}{2}\int_{\Sigma} \mathrm{Tr}\, P^2 \mathrm{d}z\mathrm{d}\bar{z} \\
&= \frac{1}{2}\int_{\Sigma} \mathrm{d}z\mathrm{d}\bar{z} \sum_{i=1}^{N} p_i^2 + \sum_{i\neq j} \epsilon^2 \Phi(z, q_j - q_i)\Phi(z, q_i - q_j) \\
& \quad \frac{1}{2}\int_{\Sigma} \mathrm{d}z\mathrm{d}\bar{z} \sum_{i=1}^{N} p_i^2 + \sum_{i\neq j} \epsilon^2 \left(\wp(z) - \wp(q_j - q_i)\right) \\
&= C \cdot \left(\frac{1}{2}\sum_{i=1}^{N} p_i^2 - \epsilon^2 \sum_{i<j} \wp(q_j - q_i) \right) + D
\end{aligned}
$$

Generalizations, research

- Can use other groups than $\mathrm{SL}(N)$: obtain interactions other than pairwise $\left\{q \mapsto q_i - q_j\right\}$
- Can start with $T^\vee G^\Sigma$ instead of $T^\vee \mathfrak{g}^\Sigma$. This is a "deformation" that is often interpreted as a relativistic generalization: the Ruijsenaars-Schneider model
- Take other moment map values: usually leads to the particles having internal degrees of freedom ("spin")
- Other kinds of reductions: start with Poisson double instead of symplectic manifold, or quasi-Hamiltonian manifold
- These reductions sometimes offer explanations of *Ruijsenaars dualities* between different systems

Main point

We have seen a *complicated* system of interacting particles that can be solved because it corresponds to a *simple* motion in the space of holomorphic principal $\mathrm{SL}(N)$-bundles over an elliptic curve.

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