Ruijsenaars-Schneider system from quasi-Hamiltonian reduction

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Main point

We will explain the integrability of the compact, trigonometric Ruijsenaars-Schneider system as a consequence of the symmetry of a much simpler dynamical system on $\mathrm{SU}(N) \times \mathrm{SU}(N)$.

The trigonometric Ruijsenaars-Schneider system

The *compact trigonometric Ruijsenaars-Schneider model* is the following:

- N particles on a circle, with coordinates $q_i \in [0, \pi)$ for $i = 1, \cdots, N$.
- Hamiltonian with $y \in (0, \pi)$ a real coupling parameter:

$$H = \sum_{i=1}^{N} \cos p_i \prod_{j \neq i} \left(1 - \frac{\sin^2 y}{\sin^2 (q_i - q_j)} \right)^{1/2}$$

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- The requirement $|q_i - q_j| \geq y$ guarantees that all square roots are real, which only has nonempty solutions if $y \leq \frac{\pi}{N}$.
- Why is this integrable? Where do the symmetries come from?

Outline

1. Simple example: Marsden-Weinstein reduction "explains" integrability in the rational Calogero-Moser system
2. Similarly, *quasi-Hamiltonian reduction* explains integrability in the current case
3. Classification of compact integrable systems allow us to describe the topology of the reduced space (it is just $\mathbf{C}P^{N-1}$)
4. New work: extension to coupling parameter values $y > \frac{\pi}{N}$

Outline

Rational Calogero-Moser system

Example

$M \subseteq T^\vee \mathbf{R}^N$, interpreted as positions and momenta of N particles in 1 dimension with center-of-mass set to 0.

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 - \sum_{i<j} \frac{\epsilon^2}{(q_i - q_j)^2}$$

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Theorem (Calogero)

This is an $2(N-1)$-dimensional integrable system, with Hamiltonians given by $H_k = \operatorname{Tr} L^k$, where L is the traceless matrix

$$L = \begin{pmatrix} p_1 & & \frac{\epsilon}{q_i - q_j} \\ & \ddots & \\ \frac{\epsilon}{q_i - q_j} & & p_N \end{pmatrix}$$

Rational Calogero-Moser system (ctd)

So some of these Hamiltonians are:

$$
\begin{aligned}
H_1 &= \sum_{i=1}^{N} p_i = 0 \\
H_2 &= \sum_{i=1}^{N} p_i^2 - \sum_{i \neq j} \frac{\epsilon^2}{(q_i - q_j)^2} \qquad (= 2H) \\
H_3 &= \sum_{i=1}^{N} p_i^3 - \sum_{i \neq j} p_i \frac{\epsilon^2}{(q_i - q_j)^2} + \sum_{i,j,k \text{ distinct}} \frac{\epsilon^3}{(q_i - q_j)(q_j - q_k)(q_k - q_i)}
\end{aligned}
$$

Rational Calogero-Moser system (ctd)

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Where do all these symmetries / conserved quantities come from?

Rational Calogero-Moser system (ctd)

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"Answer"

They exist because the motion is very simple (linear) in the matrix space.

Linear motion in a matrix space

- $G = \mathrm{SL}(N)$, $\mathfrak{g} = \mathfrak{sl}(N)$
- Phase space $M = T^\vee \mathfrak{g} = \mathfrak{g} \times \mathfrak{g}$ using Killing pairing
- Hamiltonian $H(P, Q) = \frac{1}{2} \langle P, P \rangle$

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- Phase space $M = T^\vee\mathfrak{g} = \mathfrak{g} \times \mathfrak{g}$ using Killing pairing
- Hamiltonian $H(P, Q) = \frac{1}{2}\langle P, P\rangle$
- Solution for given initial value (P_0, Q_0):

$$\begin{aligned} P(t) &= P_0 \\ Q(t) &= Q_0 + tP_0 \end{aligned}$$

Linear motion in a matrix space (ctd)

- Symmetric under adjoint action of G on $\mathfrak{g}$:
  - H and ω invariant under conjugation $P, Q \mapsto gPg^{-1}, gQg^{-1}$
  - Time evolution commutes with G-action
- Conserved quantities in involution:

$$H_k = \mathrm{Tr}\, P^k$$

- Also invariant under conjugation
- But too few: $\dim M = 2\left(n^2 - 1\right) > 2(n - 1)$

Marsden-Weinstein reduction

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- Since everything is G-invariant, we can quotient out by it.
- Hopefully, this reduces the dimension sufficiently to end up with an integrable system.

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- Since everything is G-invariant, we can quotient out by it.
- Hopefully, this reduces the dimension sufficiently to end up with an integrable system.
- But we also need to keep a non-degenerate symplectic form:
    - If we quotient out a tangent vector $\xi \in TM$, then we should also quotient out its image $\omega(\xi) \in T^\vee M$. Dually, that means restricting to a submanifold.

Marsden-Weinstein reduction (ctd)

Definition

A group action of G on M is *Hamiltonian* if its infinitesimal vector fields v_ξ for $\xi \in \mathfrak{g}$ are of the form

$$v_\xi = \{f_\xi, \cdot\}$$

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A Hamiltonian group action is *generated by a moment map* $\mu \colon M \to \mathfrak{g}^\vee$ if

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and if μ is G-equivariant.

Important example

A commuting set of Hamiltonians $h = (h_1, \cdots, h_n)$ forms a moment map for an $\mathbf{R}^n$-action, or for a $\mathbb{T}^n$-action.

Marsden-Weinstein reduction (ctd)

Theorem (Marsden, Weinstein)

If $\mu_0 \in \mathfrak{g}^\vee$ is a regular value of μ, then the space $\mu^{-1}(\mu_0)/G_{\mu_0}$ is a symplectic manifold.

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“The moment map and the G-action work together to keep the symplectic form non-degenerate”

Reduction of linear motion

Fact

$\mu(P, Q) = [P, Q] \in \mathfrak{g} \cong \mathfrak{g}^\vee$ is a moment map for the conjugation action.

Theorem

Pick the following regular value $\mu_0 \in \mathfrak{g} \cong \mathfrak{g}^\vee$:

$$\mu_0 = -\epsilon \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} + \epsilon \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

Then p, q parametrize the G_{μ_0}-equivalence classes in $\mu^{-1}(\mu_0)$ by

$$P(p, q), Q(p, q) = \begin{pmatrix} p_1 & & \frac{\epsilon}{q_i - q_j} \\ & \ddots & \\ \frac{\epsilon}{q_i - q_j} & & p_N \end{pmatrix}, \begin{pmatrix} q_1 & & 0 \\ & \ddots & \\ 0 & & q_N \end{pmatrix}$$

and they are canonical coordinates on $\mu^{-1}(\mu_0)/G_{\mu_0}$.

Recap

- Linear motion

$$t \mapsto P_0, Q_0 + tP_0$$

has $N-1$ conserved quantities in a $2(N^2-1)$ dimensional phase space.

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Recap

- Linear motion
$$t \mapsto P_0, Q_0 + tP_0$$
has $N-1$ conserved quantities in a $2(N^2-1)$ dimensional phase space.
- Is is also symmetric under conjugation $P, Q \mapsto gPg^{-1}, gQg^{-1}$
- Quotienting out and restricting yields a $2(N-1)$ dimensional space: the Calogero-Moser integrable system

Outline

Quasi-Hamiltonian manifold

- First studied by Alekseev, Malkin and Meinrenken (1998).
- A quasi-Hamiltonian manifold M has:
  - a G-action;
  - a G-equivariant map $\mu\colon M \to G$ (called moment map);
  - a G-invariant 2-form ω such that $\mathrm{d}\omega = -\frac{1}{12}\mu^*(\theta, [\theta, \theta])$, with $\theta, \bar{\theta}$ the Maurer-Cartan forms;
  - Relation between moment map and G-action:

$$\omega(v_\xi, \cdot) = \frac{1}{2}\mu^*(\theta + \bar{\theta}, \xi)$$

  - The form ω is maximally non-degenerate:

$$\ker \omega_x = \left\{ v_\xi(x) \mid \xi \in \ker \left(\mathrm{Ad}_{\mu(x)} + \mathrm{id} \right) \right\}$$

Quasi-Hamiltonian manifold (ctd)

These axioms guarantee the following:

1. To G-invariant functions h, we can associate a flow given by the vector field v_h such that $\omega(v_h, \cdot) = \mathrm{d}h$ and $\mathcal{L}_{v_h}\mu = 0$;

and we can define the *quasi-Hamiltonian reduction at* μ_0:

2. If $\mu_0 \in G$ is a regular value for μ, then $\mu^{-1}(\mu_0)/G_{\mu_0}$ is a symplectic manifold;
3. G-invariant functions with commuting flows descend to Poisson-commuting functions on the reduced space.

Comparison with Marsden-Weinstein reduction

- Moment map takes values in G instead of $\mathfrak{g}^\vee$
- Tighter relation between moment map and 2-form.
- Big phase space is not a Poisson manifold: only G-invariant Hamiltonians have an associated flow

Comparison with Marsden-Weinstein reduction

- Moment map takes values in G instead of $\mathfrak{g}^\vee$
- Tighter relation between moment map and 2-form.
- Big phase space is not a Poisson manifold: only G-invariant Hamiltonians have an associated flow
- There is a correspondence between quasi-Hamiltonian reduction with respect to G and Marsden-Weinstein reduction with respect to the loop group LG.

Big phase space

- We let

$$M = \mathrm{SU}(N) \times \mathrm{SU}(N)$$

with simultaneous conjugation action, and moment map

$$\mu(A, B) = ABA^{-1}B^{-1}$$

and 2-form

$$\omega = \langle A^{-1}\mathrm{d}A \wedge \mathrm{d}B\, B^{-1}\rangle + \langle \mathrm{d}A\, A^{-1} \wedge B^{-1}\mathrm{d}B\rangle - \langle (AB)^{-1}\mathrm{d}(AB) \wedge (BA)^{-1}\mathrm{d}(BA)\rangle$$

Reminder

The $N - 1$ simplex Δ_{N-1} are all vectors

$$\xi = (\xi_1, \cdots, \xi_N)$$

with nonnegative coefficients such that

$$\xi_1 + \cdots + \xi_N = \pi$$

Invariant functions on SU(N)

- Since SU(N) is compact, any group element is diagonalizable.
- Set of diagonal entries ordered counterclockwisely with increasing argument:

$$\left\{ \mathrm{e}^{2\mathrm{i}\gamma_i} \mid 1 \leq i \leq N \right\}$$

- We define

$$\begin{aligned} \xi_i &:= \gamma_i - \gamma_{i-1} \qquad 2 \leq i \leq N \\ \xi_1 &:= \pi + \gamma_1 - \gamma_N \end{aligned}$$

- Ambiguity: which is the first γ_i? Converse ambiguity: given ξ, can we recover the γ_i's?

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- Ambiguity: which is the first γ_i? Converse ambiguity: given ξ, can we recover the γ_i's?
- These cyclic ambiguities 'cancel' to give a correspondence between conjugacy classes in SU(N) and Δ_{N-1}

Invariant functions on SU(N) (ctd)

- Define

$$\xi \longleftrightarrow \delta(\xi)$$

for the diagonal matrix corresponding to the value $\xi \in \Delta_{N-1}$

Invariant functions on the big phase space

We define

$$\begin{aligned}\alpha(A,B) &= \xi(A)\\ H(A,B) &= \frac{1}{2}\left(\mathrm{Tr}(A)+\mathrm{Tr}(A^{\dagger})\right)\\ \beta(A,B) &= \xi(B)\end{aligned}$$

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Dynamics on the big phase space

- The flow generated by α acts on the factor B; the flow generated by β acts on the factor A.
- Explicitly, the flow generated by α_i is:

$$t \mapsto (A_0, \exp(-\mathrm{i}\nabla\alpha_i)B_0)$$

where

$$\nabla\alpha_i = E_{ii} - E_{(i+1)(i+1)}$$

if A_0 is diagonal and its entries are ordered counterclockwisely (and if the cyclic ambiguity has been resolved)
- By G-invariance of the flow, this can be extended to the whole phase space

Possibility for confusion

There is a quasi-Hamiltonian moment map

$$\mu(A, B) = ABA^{-1}B^{-1}$$

for a SU(N)-action. There are also Hamiltonians

$$\alpha, \beta \colon M \to \Delta_{N-1}$$

with associated flows.

After reduction, we will also consider β a moment map, namely for a $\mathbb{T}^{N-1}$-action.

Reduction

The moment map constraint is

$$ABA^{-1}B^{-1} = \mu_0$$

where

$$\mu_0 = \mathrm{diag}(\overbrace{e^{2iy}, \cdots, e^{2iy}}^{N-1 \text{ times}}, e^{2i(1-N)y})$$

Solving the moment map constraint

- Constraint
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- So their characteristic polynomials are equal.
- Express both in terms of $\xi(B)$ and the matrix $g(B)$ that diagonalizes B.
- (this is actually doable)

Solving the moment map constraint (ctd)

- These characteristic polynomials are equal if and only if

$$\left|g_{\ell N}(B)\right|^2 = z_\ell(\xi, y)$$

where

$$z_\ell(\xi, y) = \frac{\sin(y)^N}{\sin(Ny)} \prod_{j=1}^{N-1} \left(\cot y - \cot\left(\xi_{\sigma^\ell(1)} + \cdots + \xi_{\sigma^\ell(j)}\right)\right)$$

where we define the cyclic permutation

$$\sigma = (12 \cdots N)$$

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where we define the cyclic permutation

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- This is possible if all

$$z_\ell(\xi, y) \geq 0$$

which is a condition on the value of ξ. Its solutions are

$$\{\xi \in \Delta_{N-1} \mid \xi_i \geq y \text{ for all } i\}$$

All solutions corresponding to a given ξ

- Possible values for B are

$$B = g^{-1}\delta(\xi)g$$

if

$$\left|g_{\ell N}(B)\right|^2 = z_\ell(\xi, y)$$

- We can use the stabilizer of $\delta(\xi)$ to even set

$$g_{\ell N}(B) = \sqrt{z_\ell(\xi, y)}$$

- It turns out that all such g are in the same G_{μ_0}-orbit.

All solutions corresponding to a given ξ (ctd)

- So up to G_{μ_0}, only one possible value for B.
- Then by

$$ABA^{-1} = \mu_0 B$$

the possible values for A are parametrized by the stabilizer of B, a torus.

Recap

- We can solve the moment map constraint only if the spectral functions of B satisfy the constraints:

$$\beta(A, B) \in \{\xi \in \Delta_{N-1} \mid \xi_\ell \geq y \text{ for all } \ell\}$$

- The set $\beta^{-1}(\xi)/G_{\mu_0}$ consists of the stabilizer of B, an $N-1$-dimensional torus
- The function

$$\frac{1}{2}\left(\operatorname{Tr} A + \operatorname{Tr} A^\dagger\right)$$

is contained in the Abelian algebra generated by the α_i, so it is an integrable Hamiltonian

Explicit solutions

For given ξ satisfying the constraints, we define

$$(g_0)_{jN}(\xi) = -(g_0)_{Nj}(\xi) = \sqrt{z_j} \quad (1)$$

$$(g_0)_{NN}(\xi) = \sqrt{z_N} \quad (2)$$

$$(g_0)_{ij}(\xi) = \delta_{ij} - \frac{\sqrt{z_i z_j}}{1+\sqrt{z_N}} \quad (3)$$

and

$$L_0(\xi)_{ij} = \frac{\mathrm{e}^{\mathrm{i}y} - \mathrm{e}^{-\mathrm{i}y}}{\mathrm{e}^{\mathrm{i}y}\delta_i\delta_j^{-1} - \mathrm{e}^{-\mathrm{i}y}} \prod_{k\neq i}\left(\frac{\mathrm{e}^{\mathrm{i}y}\delta_i - \mathrm{e}^{-\mathrm{i}y}\delta_k}{\delta_i - \delta_k}\right)^{1/2} \prod_{k\neq j}\left(\frac{\mathrm{e}^{-\mathrm{i}y}\delta_j - \mathrm{e}^{\mathrm{i}y}\delta_k}{\delta_j - \delta_k}\right)^{1/2} \quad (4)$$

and

$$B = g_0^{-1}\delta(\xi)g_0$$

$$A = g_0^{-1}L_0(\xi)\Theta(p)g_0$$

Outline

Marsden-Weinstein reduction

Quasi-Hamiltonian reduction

Classification of compact integrable systems

Larger coupling parameter

Toric manifolds

Definition

A *toric manifold* is a compact symplectic manifold M of dimension $2n$ together with an effective, Hamiltonian action of the n-dimensional torus $\mathbb{T}^n$ generated by a moment map

$$\beta\colon M \to \mathrm{Lie}\,(\mathbb{T}^n)^\vee \qquad (= \mathbf{R}^n)$$

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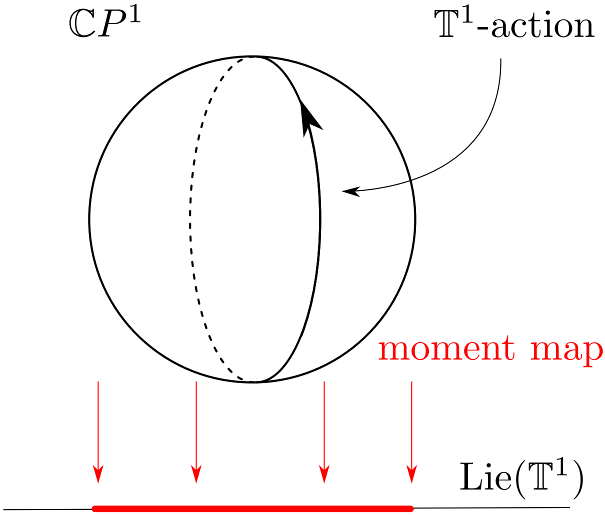
Theorem (Delzant)

Toric manifolds are classified by the image of β, and this image is always a polytope.

Example of a toric manifold



Example of a toric manifold (ctd)

- Consider $M = \mathbf{C}P^1$
- Start with $\tilde{M} = \mathbf{C}^2$ with coordinates z_0, z_1, and symplectic form

$$\tilde{\omega} = \frac{1}{2\mathrm{i}} \sum_i \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_i$$

- There is a $\mathbb{T}^2$-action.
- Diagonal $\mathbb{T}$-action has moment map

$$\tilde{\beta}(z_0, z_1) = |z_0|^2 + |z_1|^2$$

- Use Marsden-Weinstein reduction:

$$\tilde{\beta}^{-1}(\{a\})/\mathbb{T}$$

  is a symplectic manifold, and there is a residual action of $\mathbb{T}^2/\mathbb{T}$. This is $\mathbf{C}P^1$.
- The value of $a \neq 0$ determines the scale of the symplectic form.

The small phase space is a toric manifold

Recall:

- We can solve the moment map constraint only if the spectral functions of B satisfy constraints:

$$\beta(A, B) \in \{\xi \in \Delta_{N-1} \mid \xi_i \geq y \text{ for all } i\}$$

- The set $\beta^{-1}(\xi)/G_{\mu_0}$ consists of the stabilizer of B, an $N-1$-dimensional torus

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We conclude:

- Indeed, we have a $2\,(N-1)$-dimensional, compact, symplectic manifold with β a moment map. (Also, the action is effective.)

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Outline

Marsden-Weinstein reduction

Quasi-Hamiltonian reduction

Classification of compact integrable systems

Larger coupling parameter

Restriction on coupling parameter

- The restriction $y < \frac{\pi}{N}$ is natural:
  - for the physics:

$$H = \sum_{i=1}^{N} \cos p_i \prod_{j \neq i} \left(1 - \frac{\sin^2 y}{\sin^2(q_i - q_j)}\right)^{1/2}$$

    would contain imaginary square roots for larger y
  - For the reduction: the set

$$\{\xi \in \Delta_{N-1} \mid \xi_i \geq y \text{ for all } i\}$$

    tends to a single point as $y \to \frac{\pi}{N}$
- But the reduction still works for all y, as long as

$$y \neq \frac{k}{m}\pi$$

  for $2 \leq m \leq N$ and $0 \leq k \leq m$, guaranteeing that μ_0 is a regular value of μ.

Restriction on coupling parameter (ctd)

Remaining question

What happens for other values?

Restriction on coupling parameter (ctd)

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What happens for other values?

Probable answer

- For $\frac{\pi}{N} < y < \frac{\pi}{N-1}$, we obtain the set

$$\beta(A,B) \in \{\xi \in \Delta_{N-1} \mid \xi_i \leq y \text{ for all } i\}$$

- For all other y (so $\frac{\pi}{N-1} < y < \pi - \frac{\pi}{N-1}$), the moment map constraint has solutions (A,B) where β is not differentiable. (Maybe consider $\operatorname{Tr} A^k + \operatorname{Tr} A^{\dagger k}$ instead?)

Main point

We have explained the integrability of the compact, trigonometric Ruijsenaars-Schneider system as a consequence of the symmetry of a much simpler dynamical system on $\mathrm{SU}(N) \times \mathrm{SU}(N)$.

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